A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry

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Abstract. We study geodesically complete and locally compact Hadamard spaces X whose Tits boundary is a connected irreducible spherical building. We show that X is symmetric iff complete geodesics in X do not branch and a Euclidean building otherwise. Furthermore, every boundary equivalence (cone topology homeomorphism preserving the Tits metric) between two such spaces is induced by a homothety. As an application, we can extend the Mostow and Prasad rigidity theorems to compact singular (orbi)spaces of nonpositive curvature which are homotopy equivalent to a quotient of a symmetric space or Euclidean building by a cocompact group of isometries.

Contents

1 Introduction

1.1 Main result, background, motivation and an application

Hadamard manifolds are simply-connected complete Riemannian manifolds of nonpositive sectional curvature. Prominent examples are Riemannian symmetric spaces of noncompact type, but many more examples occur as universal covers of closed nonpositively curved manifolds. For instance, most Haken 3-manifolds admit metrics of nonpositive curvature [Le95]. Not the notion of sectional curvature itself, however the notion of an upper curvature bound can be expressed purely by inequalities involving the distances between finitely many points but no derivatives of the Riemannian metric, and hence generalizes from the narrow world of Riemannian manifolds to a wide class of metric spaces, cf. [Al57]. The natural generalization of Hadamard manifolds are Hadamard spaces, i.e. complete geodesic metric spaces which are nonpositively curved in the (global) sense of distance comparison, see [Ba95, KL96]. Hadamard spaces comprise besides Hadamard manifolds a large class of interesting singular spaces, among them Euclidean buildings (the discrete cousins of symmetric spaces), many piecewise Euclidean or Riemannian complexes occuring, for instance, in geometric group theory, and branched covers of Hadamard manifolds. Hadamard spaces received much attention in the last decade, notably with view to geometric group theory, a main impetus coming from Gromov's work [Gr87, Gr93].

We recall that a fundamental feature of a Hadamard space is the convexity of its distance function with the drastic consequences such as uniqueness of geodesics and in particular contractibility. This illustates that already geodesics, undoubtedly fundamental objects in geometric considerations, are rather well-behaved and their behavior can be to some extent controlled, which gets the foot in the door for a more advanced geometric understanding. The importance of the geometry of nonpositive curvature lies in the coincidence that one has a rich supply of interesting examples reaching into many different branches of mathematics (like geometric group theory,

representation theory, arithmetic) and, at the same time, these spaces share simple basic geometric properties which makes them understandable to a certain extent and in a uniform way.

We will be interested in asymptotic information and the restrictions which it imposes on the geometry of a Hadamard space X. This is related to the rigidity question, already classical in global Riemannian geometry, how topological properties of a (for instance closed) Riemannian manifold with certain local (curvature) constraints are reflected in its geometry[1] and below (1.3) we will present an application in this direction.

Let us first describe which asymptotic information we consider. The *geometric* or *ideal boundary* $\partial_\infty X$ of a Hadamard space X is defined as the set of equivalence classes of asymptotic geodesic rays.[2] The topology on X extends to a natural *cone topology* on the geometric completion $\bar{X} = X \cup \partial_\infty X$ which is compact iff X is locally compact. The ideal boundary points $\xi \in \partial_\infty X$ can be thought of as the ways to go straight to infinity[3]. It is fair to say that the topological type of $\partial_\infty X$ is not a very strong invariant, for example it is a $(n-1)$-sphere for any n-dimensional Hadamard manifold.

Besides the cone topology there is another interesting structure on $\partial_\infty X$, namely the *Tits angle metric* introduced by Gromov in full generality in [BGS85]. For two points $\xi_1, \xi_2 \in \partial_\infty X$ at infinity their Tits angle $\angle_{Tits}(\xi_1, \xi_2)$ measures the maximal visual angle $\angle_x(\xi_1, \xi_2)$ under which they can be seen from a point x inside X, or equivalently, it measures the asymptotic linear rate at which unit speed geodesic rays ρ_i asymptotic to the ideal points ξ_i diverge from each other. If X has a strictly negative curvature bound the *Tits boundary* $\partial_{Tits} X = (\partial_\infty X, \angle_{Tits})$ is a discrete metric space and only of modest interest. However, if X features substructures of extremal curvature zero, such as flats, i.e. convex subsets isometric to Euclidean space, then connected components appear in the Tits boundary and the Tits metric becomes an interesting structure.[4] The cone topology together with the Tits metric on $\partial_\infty X$ are the *asymptotic data* which we consider here. Our results find shelter under the roof of the following:

Meta-Question 1.1 *What are the implications of these asymptotic data for the geometry of a Hadamard space?*

The main result is the following characterization of symmetric spaces and Euclidean buildings of higher rank as Hadamard spaces with spherical building boundary:

[1] Since the universal cover is contractible, the entire topological information is contained in the fundamental group and one can ask which of its algebraic properties are visible in the geometry.

[2] For two unit speed geodesic rays $\rho_1, \rho_2 : [0, \infty)$ the distance $d(\rho_1(t), \rho_2(t))$ of travellers along the rays is a convex function. If it is bounded (and hence non-increasing) the rays are called *asymptotic*.

[3] Examples: The geometric completion of hyperbolic plane can be obtained by taking the closure in the Poincaré disk model; one obtains the Poincaré Compact Disk model. The geometric boundary of a metric tree is the set of its ends which is a Cantor set if it has no isolated points.

[4] $\partial_{Tits} SL(3, \mathbb{R})/SO(3)$ is the 1-dimensional spherical building associated to the real projective plane.

Main Theorem 1.2 *Let X be a locally compact Hadamard space with extendible geodesic segments*[5] *and assume that $\partial_{Tits}X$ is a connected thick irreducible spherical building. Then X is a Riemannian symmetric space or a Euclidean building.*

In the smooth case, i.e. for Hadamard manifolds, 1.2 follows from work of Ballmann and Eberlein, cf. [Eb88, Theorem B], or else from arguments of Gromov [BGS85] and Burns-Spatzier [BS87]. There is a dichotomy into two cases, according to whether geodesics in X branch or not. In the absence of branching the ideal boundaries are very symmetric because there is an involution ι_x of $\partial_\infty X$ at every point $x \in X$, and one can adapt arguments from Gromov in the proof of his rigidity theorem [BGS85]. Our main contribution lies in the case of geodesic branching. There the boundary at infinity admits in general no non-trivial symmetries and another approach is needed.

We show moreover that for the spaces considered in 1.2 the extreme situation occurs that X is completely determined by its asymptotic data up to a scale factor:

Addendum 1.3 *Let X be a symmetric space or a thick Euclidean building, irreducible and of rank ≥ 2, and let X' be another such space. Then any boundary isomorphism (cone topology homeomorphism preserving the Tits metric)*

$$\phi : \partial_\infty X \to \partial_\infty X' \tag{1}$$

is induced by a homothety.

1.3 follows from Tits classification for automorphisms of spherical buildings in the cases when X has many symmetries, e.g. when it is a Riemannian symmetric space or a Euclidean building associated to a simple algebraic group over a local field with non-archimedean valuation. This is in particular true if $rank(X) \geq 3$ however it does not cover the cases when X is a rank 2 Euclidean building with small isometry group. Our methods provide a uniform proof in all cases and in particular a direct argument in the symmetric cases.

A main motivation for us was Mostow's Strong Rigidity Theorem for locally symmetric spaces, namely the irreducible case of higher rank:

Theorem 1.4 ([Mos73]) *Let M and M' be locally symmetric spaces whose universal covers are irreducible symmetric spaces of rank ≥ 2. Then any isomorphism $\pi_1(M) \to \pi_1(M')$ of fundamental groups is induced by a homothety $M \to M'$.*

It is natural to ask whether locally symmetric spaces are rigid in the wider class of closed manifolds of nonpositive sectional curvature. This is true and the content of Gromov's Rigidity Theorem [BGS85]. As an application of our main results we present an extension of Mostow's theorem as well as Prasad's analogue for compact quotients of Euclidean buildings [Pra79] to the larger class of singular nonpositively curved (orbi)spaces:

Application 1.5 *Let X be a locally compact Hadamard space with extendible geodesic segments and let X_{model} be a symmetric space (of noncompact type) or a thick Euclidean building. Suppose furthermore that all irreducible factors of X_{model} have rank*

[5] I.e. every geodesic segment is contained in a complete geodesic.

≥ 2. *If the same finitely generated group Γ acts cocompactly and properly discontinuously on X and X_{model} then, after suitably rescaling the metrics on the irreducible factors of X_{model}, there is a Γ-equivariant isometry $X \to X_{model}$.*

I.e. among (possibly singular) geodesically complete compact spaces of nonpositive curvature (in the local sense), quotients of irreducible higher rank symmetric spaces or Eulidean buildings are determined by their homotopy type.

Example 1.6 *On a locally symmetric space with irreducible higher rank universal cover there exists no piecewise Euclidean singular metric of nonpositive curvature.*

As we said, 1.5 is due to Gromov [BGS85] in the case that X is smooth Riemannian. Although we extend Gromov's extension of Mostow Rigidity further to singular spaces, the news of 1.5 lie mainly in the building case.

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I make extensive use of the geometric approach to the theory of generalized Tits buildings developed in [KL96] together with Bruce Kleiner and thank him for numerous discussions related to the subject of this paper. I am grateful to Werner Ballmann and Misha Kapovich for pointing out countably many errors in a late version of the manuscript, to Jens Heber for communicating a reference and to the Complutense library in Madrid for several crucial suggestions. Part of this work was done while visiting the Instituto de Matematicas de la UNAM, Unidad Cuernavaca, in Mexico. I thank the institute for its great hospitality, especially Pepe Seade and Alberto Verjovski.

1.2 Around the argument

In this section I attempt to describe the scenery around the proofs of 1.2 and 1.3 for the rank 2 case, i.e. when maximal flats in X have dimension 2 and $\partial_{Tits}X$ is 1-dimensional. The rank 2 case is anyway the critical case because there the rigidity is qualitatively weaker than in the case of rank ≥ 3. This difference is reflected in Tits classification theorem for spherical buildings [Ti74] which asserts, roughly speaking, that all thick irreducible spherical buildings of rank ≥ 3 (that is, dimension ≥ 2) are canonically attached to simple algebraic or classical groups. In contrast there exist uncountably many absolutely asymmetric 1-dimensional spherical buildings, for example those corresponding to exotic projective planes, and uncountably many of them occur as Tits boundaries of rank 2 Euclidean buildings with trivial isometry group.

For a singular geodesic l in X, i.e. a geodesic asymptotic to vertices in $\partial_{Tits}X$, we consider the union $P(l)$ of all geodesics parallel to l and its cross section $CS(l)$ which is a locally compact Hadamard space with discrete Tits boundary (as for a rank 1 space).[6] For two asymptotic geodesics l and l' one can canonically identify the ideal

[6] For instance, if $X = SL(3,\mathbb{R})/SO(3)$ then the cross sections of singular geodesics are hyperbolic planes. More generally, if X is a symmetric space of rank 2 then these cross sections are rank-1 symmetric spaces. If X is a Euclidean building of rank 2 they are rank-1 Euclidean buildings, i.e. metric trees.

boundaries $\partial_\infty CS(l)$ and $\partial_\infty CS(l')$. A priori this identification of boundaries does not extend to an isometry between the cross sections, but it extends to an isometry between their "convex cores", i.e. between closed convex subsets $C \subseteq CS(l)$ and $C' \subseteq CS(l')$ which are minimal among the closed convex subsets satisfying $\partial_\infty C = \partial_\infty CS(l)$ and $\partial_\infty C' = \partial_\infty CS(l')$. Due to a basic rigidity phenomenon in the geometry of nonpositively curved spaces, the so-called "Flat Strip Theorem", the convex cores are unique up to isometry. We see that, essentially due to the connectedness of the Tits boundary, there are many natural identifications between the various parallel sets and observe that, by composing them, one can generate large groups of isometries acting on the cores C_l of the cross sections (see section 3). We denote the closures of these "holonomy" subgroups in $Isom(C_l)$ by $Hol(l)$. They are large in the sense that $Hol(l)$ acts 2-fold transitively on $\partial_\infty CS(l)$.[7] Hence, *however unsymmetric X itself may be, the cross sections of its parallel sets are always highly symmetric,* and this is the key observation at the starting point of our argument.

The high symmetry imposes a substantial restriction on the geometry of the cross sections and the major step in our proof of 1.2 is a rank-1 analogue for spaces with high symmetry:

Theorem 1.7 *Let Y be a locally compact Hadamard space with extendible rays and at least 3 points at infinity. Assume that Y contains a closed convex subset C with full ideal boundary $\partial_\infty C = \partial_\infty Y$ so that $Isom(C)$ acts 2-fold transitively on $\partial_\infty C$. Then the following dichotomy occurs:*

1. If some complete geodesics in Y branch then Y is isometric to the product of a metric tree (with edges of equal length) and a compact Hadamard space.

2. If complete geodesics in Y do not branch then there exists a rank-1 Riemannian symmetric model space Y_{model}, and a boundary homeomorphism $\partial_\infty Y \to \partial_\infty Y_{model}$ carrying $Isom_o(C)$ to $Isom_o(Y_{model})$.[8]

In particular, the ideal boundary of every cross section is homeomorphic to a sphere, a Cantor set or a finite set of cardinality ≥ 3.

As we explained, the geometry of X is rigidified by the various identifications between cores of cross sections of parallel sets. This can be nicely built in the picture of the geometric compactification of X as follows: We mentioned that the convex cores of the cross sections $CS(l)$ for all lines l asymptotic to the same vertex $\xi \in \partial_{Tits} X$ can be canonically identified to a Hadamard space C_ξ. It has rank 1 in the sense that it satisfies the visibility property, or equivalently, its Tits boundary is discrete. $\partial_\infty C_\xi$ can be reinterpreted as the compact topological space of Weyl chambers (arcs) emanating from the vertex ξ. One can now blow up the geometric boundary $\partial_\infty X$ by replacing each vertex ξ by the geometric compactification $\bar{C}_\xi$ and gluing the endpoints of Weyl arcs to the corresponding boundary points in $\partial_\infty C_\xi$. This generalizes a construction

[7] If X is a rank-2 symmetric space, $Hol(l)$ contains the identity component of the isometry group of the rank-1 symmetric space $CS(l)$. So in the example $X = SL(3,\mathbb{R})/SO(3)$ ($X = SL(3,\mathbb{C})/SU(3)$) the action of $Hol(l)$ on the boundary of the hyperbolic plane (hyperbolic 3-space) $CS(l)$ is even 3-fold transitive (by Möbius transformations). More generally the action is 3-fold transitive if $\partial_{Tits} X$ is the spherical building associated to an (abstract) projective plane.

[8] It seems unclear whether in this case one should not be able to find an embedded rank-1 symmetric space inside Y.

by Karpelevič for symmetric spaces [Ka]. We denote the resulting refined boundary by $\partial_\infty^{fine}X$, and by $\partial_\infty^{fine,\partial}X$ the part which one obtains by inserting only the boundaries $\partial_\infty C_\xi$ instead of the full compactifications $\bar{C}_\xi$. *The rigidity expresses itself in the action of the holonomy groupoid which appears on the blown up locus of the refined boundary $\partial_\infty^{fine}X$ due to the connectedness of $\partial_{Tits}X$:* For any two antipodal vertices $\xi_1, \xi_2 \in \partial_{Tits}X$, i.e. vertices of Tits distance $\angle_{Tits}(\xi_1, \xi_2) = \pi$, there is a canonical isometry

$$C_{\xi_1} \leftrightarrow C_{\xi_2} \tag{2}$$

because the spaces C_{ξ_i} embed as minimal convex subsets into the cross section $CS(\{\xi_1, \xi_2\})$ of the family of parallel geodesics asymptotic to ξ_1, ξ_2. We can compose such isometries hopping along finite sequences of successive antipodes. For any two vertices $\xi, \eta \in \partial_{Tits}X$ we denote by $Hol(\xi, \eta) \subseteq Isom(C_\xi, C_\eta)$ the closure of the subset of all isometries $C_\xi \to C_\eta$ which arise as finite composites of isometries (2) (cf. section 3). In particular, the holonomy groups $Hol(\xi) := Hol(\xi, \xi)$ act on the inserted spaces $\bar{C}_\xi$. These actions can be thought of as an additional geometric structure on the spaces $\partial_\infty C_\xi$, namely as the analogue of a conformal structure; for instance if $\partial_\infty C_\xi$ is homeomorphic to a sphere then due to 1.7 it can be identified with the boundary of a rank-1 symmetric space up to conformal diffeomorphism.

Comment on the proof of 1.2: It is easy to see that all cross sections $CS(l)$ have extendible geodesic rays if X is geodesically complete (5.3).

If some complete geodesics branch in X then there is a cross section $CS(l)$ with branching geodesics (5.6) and, apparently less trivially to verify, even all cross sections have this property (5.29). The rank-1 result 1.7 then implies that the cross sections of all parallel sets are metric trees (up to a compact factor). From this point it is fairly straight-forward to conclude in one way or another that X is a Euclidean building (section 5.4.3).

If complete geodesics in X do not branch we can adapt arguments of Gromov from the proof of his Rigidity Theorem [BGS85]. The reflections at points $x \in X$ give rise to involutive automorphisms $\iota_x : \partial_\infty X \to \partial_\infty X$ of the topological spherical building $\partial_\infty X$. One obtains a proper map $X \hookrightarrow Aut(\partial_\infty X)$ into the group of boundary automorphisms and hence finds oneself in the situation that the topological spherical building $\partial_{Tits}X$ is highly symmetric. (It satisfies the so-called Moufang property.) $Aut(\partial_\infty X)$ is a locally compact topological group [BS87]. Similar to [BS87], establishing transitivity and contraction properties for the dynamics of $Aut(\partial_\infty X)$ on $\partial_\infty X$ allows to show, using a deep result by Gleason and Yamabe on the approximation of locally compact topological groups by Lie groups, that $Aut(\partial_\infty X)$ is a semisimple Lie group and the isometry group of a Riemannian symmetric model space X_{model}. The involutions ι_x can be characterized as order 2 elements with compact centralizer and hence correspond to point reflections in X_{model}. One obtains a map $\Phi : X \to X_{model}$ which is clearly affine in the sense that it preserves flats. It immediately follows that Φ is a homothety, concluding the proof of 1.2.

Comment on the proof of 1.3: Any boundary isomorphism (1) has continuous differentials

$$\Sigma_\xi \phi : \Sigma_\xi \partial_{Tits}X \to \Sigma_{\phi\xi} \partial_{Tits}X' \tag{3}$$

and hence lifts to a map

$$\partial_\infty^{fine,\partial}X \to \partial_\infty^{fine,\partial}X' \tag{4}$$

of partially refined boundaries. The differentials (3) are conformal in the sense that they preserve the holonomy action, i.e. the induced homeomorphisms

$$Homeo(\partial_\infty C_\xi, \partial_\infty C_\eta) \to Homeo(\partial_\infty C_{\phi\xi}, \partial_\infty C_{\phi\eta})$$

carry the holonomy groupoid Hol^X to the holonomy groupoid $Hol^{X'}$. This sets us on the track towards the proof of 1.3: After proving 1.2 we may assume that X is a symmetric space or a Euclidean building. Then the C_ξ are rank-1 symmetric spaces or metric trees, respectively, and the differentials $\Sigma_\xi\phi$ actually extend to homotheties $C_\xi \to C_{\phi\xi}$. This means that the lift (4) of ϕ improves to a holonomy equivariant map

$$\partial_\infty^{fine} X \longrightarrow \partial_\infty^{fine} X' \tag{5}$$

between the full refined geometric boundaries. Since points in the blow ups C_ξ are equivalence classes of strongly asymptotic geodesics, (5) encodes a correspondence between singular geodesics in X and X'. If X is a Euclidean building then this can be used in a final step to set up a correspondence between vertices which preserves apartments and extends to a homothety $X \to X'$, hence concluding the proof of 1.3 in this case (section 5.5) . If X is a symmetric space then 1.3 already follows from the arguments in the proof of 1.2.

The paper is desorganized as follows: In section 2 we discuss preliminaries. In particular we establish the existence of convex cores for Hadamard spaces under fairly general conditions (section 2.1.2) and introduce the spaces of strong asymptote classes which will serve as an important tool in the construction of the holonomy groupoid. The holonomy groupoid is discussed in section 3 where we explain the symmetries of parallel sets. In section 4 we prove the rigidity results for "rank 1" spaces with high symmetry and in section 5 the main results for higher rank spaces.

2 Preliminaries

2.1 Hadamard spaces

For basics on Hadamard spaces and, more generally, spaces with curvature bounded above we refer to the first two chapters of [Ba95] and section 2 of [KL96]. Spaces of directions and Tits boundaries are discussed there and it is verified that they are CAT(1) spaces. Let us emphasize that we mean by the *Tits boundary* $\partial_{Tits}X$ of the Hadamard space X the geometric boundary $\partial_\infty X$ equipped with the *Tits angle metric* $\angle_{Tits}$ and not with the associated path metric[9].

In the following paragraphs we supply a few auxiliary facts needed later in the text.

2.1.1 Filling spheres at infinity by flats

The following result generalizes an observation by Schroeder in the smooth case, cf. [BGS85].

[9] If $\partial_{Tits}X$ is a spherical building then it has diameter π with respect to the path metric and hence the path metric coincides with $\angle_{Tits}$.

Proposition 2.1 *Let X be a locally compact Hadamard space and let $s \subseteq \partial_{Tits} X$ be a unit sphere which does not bound a unit hemisphere in $\partial_{Tits} X$. Then there exists a flat $F \subseteq X$ with $\partial_\infty F = s$.*

Proof: Let s be isometric to the unit sphere of dimension $d \geq 0$ and pick $d+1$ pairs of antipodes $\xi_0^\pm, \ldots, \xi_d^\pm$ so that

$$\angle_{Tits}(\xi_i^\pm, \xi_j^\pm) = \pi/2 \qquad \text{and} \qquad \angle_{Tits}(\xi_i^\pm, \xi_j^\mp) = \pi/2 \tag{6}$$

for all $i \neq j$. If for some point $x \in X$ and some index i holds $\angle_x(\xi_i^+, \xi_i^-) = \pi$ then the union $X' = P(\{\xi_i^+, \xi_i^-\})$ of geodesics asymptotic to $\xi_i^\pm$ is non-empty and s determines a $(d-1)$-sphere $s' \subseteq \partial_{Tits} X'$ which does not bound a unit hemisphere. Moreover any flat $F' \subseteq X'$ filling s' determines a flat F filling s and we are reduced to the same question with one dimension less. We can hence proceed by induction on the dimension d and the claim follows if we can rule out the situation that

$$\angle_x(\xi_i^+, \xi_i^-) < \pi \tag{7}$$

holds for all x and i. In this case we obtain a contradiction as follows. Assume first that for some (and hence any) point $x_0 \in X$ the intersection of the horoballs $Hb(\xi_i^\pm, x_0)$ is unbounded and thus contains a complete geodesic ray r. The ideal endpoint $\eta \in \partial_\infty X$ of r satisfies $\angle_{Tits}(\eta, \xi_i^\pm) \leq \pi/2$ because the Busemann functions $B_{\xi_i^\pm}$ monotonically non-increase along r. By the triangle inequality follows $\angle_{Tits}(\eta, \xi_i^\pm) = \pi/2$ because $\xi_i^\pm$ are antipodes. The CAT(1) property of $\partial_{Tits} X$ then implies that there is a unit hemisphere $h \subseteq \partial_{Tits} X$ with center η and boundary s, but this contradicts our assumption. Therefore the intersection of the horoballs $Hb(\xi_i^\pm, x_0)$ is compact for all $x_0 \in X$ and the convex function $\max B_{\xi_i^\pm}$ is proper and assumes a minimum in some point x. Denote by $r_i^\pm : [0, \infty) \to X$ the ray with $r_i^\pm(0) = x$ and $r_i^\pm(\infty) = \xi_i^\pm$. (6) implies that $B_{\xi_i^\pm}$ non-increases along $r_j^\pm$ for $i \neq j$. Hence, if x_j denotes the midpoint of the segment $\overline{r_j^+(1) r_j^-(1)}$ then $B_{\xi_i^\pm}(x_j) \leq B_{\xi_i^\pm}(x)$ for all i and, by (7), $B_{\xi_j^\pm}(x_j') < B_{\xi_j^\pm}(x)$ for some point $x_j' \in \overline{x x_j}$. This means that by replacing x we can decrease the values of one pair of Busemann functions while not increasing the others. By iterating this procedure at most $d+1$ times we find a point x' with $\max B_{\xi_i^\pm}(x') < \max B_{\xi_i^\pm}(x)$, a contradiction. $\square$

2.1.2 Convex cores

For a subset $A \subseteq \partial_\infty Y$ we denote by $\mathcal{C}_A$ the family of closed convex subsets $C \subseteq Y$ with $\partial_\infty C \supseteq A$. $\mathcal{C}_A$ is non-empty, partially ordered and closed under intersections.

Proposition 2.2 *Let Y be a locally compact Hadamard space.*

1. Suppose that $s \subseteq A \subseteq \partial_\infty Y$ and s is a unit sphere with respect to the Tits metric which does not bound a unit hemisphere. Then $\mathcal{C}_A$ contains a minimal element.

2. Suppose that $A \subseteq \partial_\infty Y$ so that $\mathcal{C}_A$ has minimal elements. Then the union Y_0 of all minimal elements in $\mathcal{C}_A$ is a convex subset of Y. It decomposes as a metric product

$$Y_0 \cong C \times Z \tag{8}$$

where Z is a compact Hadamard space and the layers $C \times \{z\}$ are precisely the minimal elements in $\mathcal{C}_A$.

Proof: According to 2.1, there exists a non-empty family of flats in Y with ideal boundary s and the family is compact because otherwise s would bound a unit hemisphere. The union $P(s)$ of these flats is a convex subset of Y.

Sublemma 2.3 *Let F be a flat and C a closed convex subset in Y so that $\partial_\infty F \subseteq \partial_\infty C$. Then C contains a flat F' parallel to F.*

Proof: For any points $x \in C$ and $y \in F$ there is a point $x' \in C$ so that $d(x', y) \leq d(x, F)$. Hence there exists a point $x'' \in C$ which realizes the nearest point distance of F and C: $d(x'', F) = d(C, F)$. Then the union of rays emanating from x'' and asymptotic to points in $\partial_\infty F$ forms a flat F' parallel to F. $\square$

Hence every convex subset $C \in \mathcal{C}_A$ intersects $P(s)$ in a non-empty compact family of flats and therefore determines a non-empty compact subset $U(C)$ in the compact cross section $CS(s)$ (compare definition 3.4). We order the sets $C \in \mathcal{C}_A$ by inclusion and observe that the assignment $C \mapsto U(C)$ preserves inclusion.

Sublemma 2.4 *Let (S_ι) be an ordered decreasing family of non-empty compact subsets of a compact metric space Z. Then the intersection of the S_ι is not empty.*

Proof: For every $n \in \mathbb{N}$ we can cover Z by finitely many balls of radius $1/n$ and therefore there exists a ball $B_{1/n}(z_n)$ which intersects all sets S_ι. Any accumulation point of the sequence (z_n) is contained in the intersection of the S_ι. $\square$

Any decreasing chain of sets $C_\iota \in \mathcal{C}_A$ yields a decreasing chain of compact cross sections $U(C_\iota)$ and hence has non-empty intersection. It follows that $\emptyset \neq \bigcap C_\iota \in \mathcal{C}_A$ and, by Zorn's lemma or otherwise, we conclude that $\mathcal{C}_A$ contains a minimal non-empty subset.

Now let $C_1, C_2 \in \mathcal{C}_A$ be minimal. For any $y_1 \in C_1$ the closed convex subset $\{y \in C_1 : d(y, C_2) \leq d(y_1, C_2)\}$ of C_1 contains A in its ideal boundary and, by minimality of C_1, is all of C_1. It follows that $d(\cdot, C_2)$ is constant on C_1 and the nearest point projection $p_{C_2C_1} : C_1 \to C_2$ is an isometry. For a decomposition $d(C_1, C_2) = d_1 + d_2$ as a sum of positive numbers, the set $\{y \in Y : d(y, C_i) = d_i \text{ for } i = 1, 2\}$ is a minimal element in $\mathcal{C}_A$. Hence Y_0 is convex.

Sublemma 2.5 *For minimal elements $C_1, C_2, C_3 \in \mathcal{C}_A$ the self-isometry $\psi = p_{C_1C_2} \circ p_{C_2C_3} \circ p_{C_3C_1}$ of C_1 is the identity.*

Proof: ψ preserves the central flat f in C_1 with ideal boundary s. Furthermore, $\psi\big|_f$ preserves all Busemann functions centered at ideal points $\in s$. Thus ψ restricts to the identity on f. Since $\partial_\infty \psi = id$ and C_1 is minimal it follows that ψ fixes C_1 pointwise. $\square$

Now choose a minimal set $C \in \mathcal{C}_A$ and a point $y \in C$. Then the set Z of points $p_{C'C}(y)$, where C' runs through all minimal elements in $\mathcal{C}_A$, is convex. It is easy to see that Y_0 is canonically isometric to $C \times Z$. Z must be compact because $CS(s)$ is. This concludes the proof of 2.2. $\square$

The compact Hadamard space Z in (8) has a well-defined center z_0. We call the layer $C \times \{z_0\}$ the *central* minimal convex subset in $\mathcal{C}_A$.

Definition 2.6 *If $\mathcal{C}_{\partial_\infty Y}$ has minimal elements then the* **convex core** *$core(Y)$ of Y is defined as the central minimal closed convex subset in $\mathcal{C}_{\partial_\infty Y}$.*

If the convex core exists it is preserved by all isometries of Y.

Lemma 2.7 *Let Y be a locally compact Hadamard space which has a convex core. If $core(Y)$ has no Euclidean factor then any isometry with trivial action at infinity fixes $core(Y)$ pointwise.*

Proof: Let ϕ be an isometry which acts trivially at infinity. Then its displacement function is constant on the central convex subset C. It is zero because C does not split off a Euclidean factor. $\square$

2.1.3 Spaces of strong asymptote classes

Let X be a Hadamard space. For a point $\xi \in \partial_\infty X$ let us consider the rays asymptotic to ξ. The *asymptotic distance* of two rays $\rho_i : [0, \infty) \to X$ is given by their nearest point distance

$$d_\xi(\rho_1, \rho_2) = \inf_{t_1, t_2 \to \infty} d(\rho_1(t_1), \rho_2(t_2)), \tag{9}$$

which equals $\lim_{t\to\infty} d(\rho_1(t), \rho_2(t))$ when the rays are parametrized so that $B_\xi \circ \rho_1 \equiv B_\xi \circ \rho_2$. We call the rays ρ_i *strongly asymptotic* if their asymptotic distance is zero. The asymptotic distance (9) defines a metric on the space X^*_ξ of strong asymptote classes.

Proposition-Definition 2.8 *The metric completion X_ξ of X^*_ξ is a Hadamard space.*

Proof: Any two points in X^*_ξ are represented by rays $\rho_1, \rho_2 : [0, \infty) \to X$ asymptotic to ξ and initiating on the same horosphere centered at ξ. Denote by $\mu_s : [s, \infty) \to X$ the ray asymptotic to ξ whose starting point $\mu_s(s)$ is the midpoint of $\overline{\rho_1(s)\rho_2(s)}$. The triangle inequality implies that $d(\rho_1(t), \mu_s(t)) + d(\mu_s(t), \rho_2(t)) - d(\rho_1(t), \rho_2(t)) \leq d(\rho_1(s), \rho_2(s)) - d(\rho_1(t), \rho_2(t)) \to 0$ as $s, t \to \infty$ with $s \leq t$. Hence $d(\mu_s(t), \mu_t(t)) \to 0$ and $d_\xi(\mu_s, \mu_t) \to 0$, i.e. (μ_s) is a Cauchy sequence and its limit in X_ξ is a midpoint for $[\rho_1]$ and $[\rho_2]$. In this manner we can assign to every pair of points $[\rho_1], [\rho_2] \in X^*_\xi$ a well-defined midpoint $m \in X_\xi$. If $[\rho'_1], [\rho'_2] \in X^*_\xi$ is another pair of points so that $d([\rho_i], [\rho'_i]) \leq \delta$ then $d(m, m') \leq \delta$. It follows that there exist midpoints for all pairs of points in X_ξ. As a consequence, any two points in X_ξ can be connected by a geodesic.

Any finite configuration $\mathcal{F}$ of points in X^*_ξ corresponds to a finite set of rays $\rho_i : [0, \infty) \to X$ asymptotic to ξ and synchronized so that for any time t the set $\mathcal{F}_t$ of points $\rho_i(t)$ lies on one horosphere centered at ξ. The finite metric spaces $(\mathcal{F}_t, d_X)$ Hausdorff converge to $(\mathcal{F}, d_\xi)$ and hence distance comparison inequalities are inherited. It follows that geodesic triangles satisfy the CAT(0) comparison inequality. $\square$

We will also X_ξ call the *space of strong asymptote classes* at $\xi \in \partial_\infty X$. It had been considered by Karpelevič in the case of symmetric spaces, see [Ka].

2.1.4 Types of isometries

We recall the standard classification of isometries into axial, elliptic and parabolic ones: For any isometry ϕ of a Hadamard space X its displacement function $\delta_\phi : x \mapsto d(x, \phi x)$ is convex. ϕ is called semisimple if δ_ϕ attains its infimum. There are two types of semisimple isometries: ϕ is *elliptic* if the minimum is zero and has fixed points in this case. If the minimum is strictly positive then ϕ is *axial* and there is a non-empty family of ϕ-invariant parallel geodesics, the *axes* of ϕ. If δ_ϕ does not have a minimum then ϕ is called *parabolic*. The fixed point set of a parabolic isometry in $\partial_{Tits}X$ is non-empty and contained in a closed ball of radius $\pi/2$.

Definition 2.9 *For $\xi \in \partial_\infty X$ we define the* **parabolic stabilizer** *P_ξ as the group consisting of all elliptic and parabolic isometries which preserve every horosphere centered at ξ.*

Note that there are parabolic isometries which fix more than one point at infinity and do not preserve the horospheres centered at some of their ideal fixed points.

Definition 2.10 *An isometry ϕ of a locally compact Hadamard space X is called* **purely parabolic** *iff its conjugacy class accumulates at the identity. If $Isom(X)$ is cocompact then this is equivalent to the property that for every $\delta > 0$ there exist arbitrarily large balls on which the displacement of ϕ is $\leq \delta$.*

2.2 Visibility Hadamard spaces

Let Y be a locally compact Hadamard space with at least 3 ideal boundary points. We assume that the Tits metric on $\partial_\infty Y$ is discrete, or equivalently, that Y enjoys the *visibility property* introduced in [EO73]: any two points at infinity are ideal endpoints of some complete geodesic. Then any two distinct ideal boundary points ξ and η have Tits distance π and the family of (parallel) geodesics asymptotic to ξ, η is non-empty and compact; we denote their union by $P(\{\xi, \eta\})$. The visibility property is clearly inherited by closed convex subsets. The terminology *visibility* is motivated by the following basic fact:

Lemma 2.11 *For every $y \in Y$ and every $\epsilon > 0$ there exists $R > 0$ such that the following is true: If $\overline{pq}$ is a geodesic segment not intersecting the ball $B_R(y)$ then $\angle_y(p, q) \leq \epsilon$.*

Proof: See [EO73]. $\square$

Consequence 2.12 *Let A be a compact subset of $\partial_\infty Y \times \partial_\infty Y \setminus Diag$. Then the set of all geodesics $c \subset Y$ satisfying $(c(-\infty), c(\infty)) \in A$ is compact.*

Proof: This set B of geodesic is certainly closed. If B would contain an unbounded sequence of geodesics c_n then the corresponding sequence of points $(c_n(-\infty), c_n(\infty))$ in A would accumulate at the diagonal Δ, contradicting compactness. $\square$

Remark 2.13 *Visibility Hadamard spaces with cocompact isometry group are large-scale hyperbolic in the sense of Gromov.*

A sequence $(\phi_n) \subset P_\xi$ diverges to infinity, $\phi_n \to \infty$, iff ϕ_n converges to the constant map with value ξ uniformly on compact subsets of $\partial_\infty Y \setminus \{\xi\}$.

Lemma 2.14 *Assume that for different ideal points $\xi, \eta \in \partial_\infty Y$ there are sequences of parabolics $\phi_n \in P_\xi$ and $\psi_n \in P_\eta$ diverging to infinity. Then $\phi_n\psi_n$ is axial for large n.*

Proof: Let U and V be disjoint neighborhoods of ξ, η respectively. Then $\phi_n^{\pm1}(\partial_\infty Y \setminus U) \subset U$ and $\psi_n^{\pm1}(\partial_\infty Y \setminus V) \subset V$ for large n which implies

$$\alpha_n(\partial_\infty Y \setminus V) \subset U \qquad \text{and} \qquad \alpha_n^{-1}(\partial_\infty Y \setminus U) \subset V \tag{10}$$

with $\alpha_n = \phi_n\psi_n$. α_n can't be elliptic (for large n) because then $(\partial_\infty \alpha_n^k)_{k\in\mathbb{N}}$ would subconverge to the identity, contradicting (10). α_n can't be parabolic either because then $(\partial_\infty \alpha_n^k)_{k\in\mathbb{N}}$ would converge to a constant function every where pointwise, which is also excluded by (10). Therefore α_n is axial for large n. $\square$

2.3 Buildings: Definition, vocabulary and examples

A geometric treatment of spherical and Euclidean Tits buildings within the framework of Aleksandrov spaces with curvature bounded above has to some extent been carried through in [KL96]. We will use these results and for the convenience of the reader we briefly recall some of the basic definitions and concepts.

2.3.1 Spherical buildings

A *spherical Coxeter complex* consists of a unit sphere S and a finite *Weyl group* $W \subset Isom(S)$ generated by reflections at *walls*, i.e. totally geodesic subspheres of codimension 1. The walls divide S into open convex subsets whose closures are the *chambers*. These are fundamental domains for the action of W on S and project isometrically to the orbit space, the model Weyl chamber $\Delta_{model} = W\backslash S$. A *panel* is a codimension-1 face of a chamber.

A *spherical building* modelled on the Coxeter complex (S, W) is a CAT(1) space[10] B together with an atlas of charts, i.e. isometric embeddings $\iota : S \hookrightarrow B$. The image of a chart is an *apartment* in B. We require that any two points are contained in an apartment and that the coordinate changes between charts are induced by isometries in W. The notions of wall, chamber, panel etc. transfer from the Coxeter complex to the building. There is a canonical 1-Lipschitz continuous *accordeon map* $\theta_B : B \to \Delta_{model}$ folding the building onto the model chamber so that every chamber projects isometrically. $\theta_B\xi$ is called the *type* of a point $\xi \in B$. ξ is *regular* if it lies in the interior of a chamber.

B is *thick* if every panel is adjacent to at least 3 chanbers. If B has no spherical de Rham factor, i.e. if W acts without fixed points, then the chambers are simplices and

[10] A *CAT(1) space* is a complete geodesic metric space with upper curvature bound 1 in the sense of Aleksandrov.

B carries a natural structure of a piecewise spherical simplicial complex. In this case we'll call the faces also simplices. A thick spherical building B is called *irreducible* if the corresponding linear representation of W is irreducible. This is equivalent to the assertions that B does not decompose as a spherical join, and that Δ_{model} does not decompose.

Tits originally introduced buildings to invert Felix Kleins Erlanger Programm and to provide geometric interpretations for algebraic groups, i.e. to construct geometries whose automorphism groups are closely related to these groups. The simplest interesting examples of irreducible spherical buildings are the buildings associated to projective linear groups. In dimension 1, one can more generally construct a spherical building for every abstract projective plane, possibly with trivial group of projective transformations:

Example 2.15 *Given an abstract projective plane $\mathcal{P}$ one constructs the corresponding 1-dimensional irreducible spherical building $B(\mathcal{P})$ as follows. There are two sorts of vertices in $B(\mathcal{P})$: red vertices corresponding to points in $\mathcal{P}$ and blue vertices corresponding to lines. One draws an edge of length $\pi/3$ between a red and a blue vertex iff they are incident. The edges in $B(\mathcal{P})$ correspond to lines in $\mathcal{P}$ with a marked point. The apartments in $B(\mathcal{P})$, i.e. closed paths of length 2π and consisting of 6 edges, correspond to tripels of points (respectively lines) in general position. From the incidence properties of projective planes one easily deduces that any two edges are contained in an apartment and that there are no closed paths of length $< 2\pi$, i.e. $B(\mathcal{P})$ is a CAT(1) space.*

Of course, a topological projective plane yields a topological spherical building.

Remark 2.16 (Exotic smooth projective planes) *As Bruce Kleiner pointed out one can produce exotic (smooth) projective planes by perturbing a smooth projective plane, for instance one of the standard projective planes $P\mathbb{R}^2$, $P\mathbb{C}^2$ or $P\mathbb{H}^2$.*

2.3.2 Euclidean buildings

A *Euclidean Coxeter complex* consists of a Euclidean space E and an *affine Weyl group* $W_{aff} \subset Isom(E)$ generated by reflections at *walls*, i.e. affine subspaces of codimension 1, so that the image W of W_{aff} in $Isom(\partial_{Tits}E)$ is a finite reflection group and $(\partial_{Tits}E, W)$ thus a spherical Coxeter complex.

A *Euclidean building* is a Hadamard space X with the following additional structure: There is a canonical maximal atlas of isometric embeddings $\iota : E \hookrightarrow X$ called *charts* so that the coordinate changes are induced by isometries in W_{aff}. Any geodesic segment, ray and complete geodesic is contained in an *apartment*, i.e. the image of a chart. The charts assign to any non-degenrate segment $\overline{xy}$ a well-defined *direction* $\theta(\overline{xy})$ in the *anisotropy polyhedron* Δ_{model}, the model Weyl chamber of $(\partial_{Tits}E, W)$. We request that for any two non-degenerate segments $\overline{xy}$ and $\overline{xz}$ the angle $\angle_x(y, z)$ takes one of the finitely many values which can occur in $(\partial_{Tits}E, W)$ as distance between a point of type $\theta(\overline{xy})$ and a point of type $\theta(\overline{xz})$. (This is called the *angle rigidity property* in [KL96].)

The *rank* of X is $dim(E)$. The spaces of directions $\Sigma_x X$ and the Tits boundary $\partial_{Tits}X$ inherite canonical spherical building structures modelled on $(\partial_{Tits}E, W)$. X

is *thick (irreducible)* if $\partial_{Tits}X$ is thick (irreducible). X is called *discrete* if W_{aff} is a discrete subgroup of $Isom(E)$. Thick locally compact Euclidean buildings are discrete and they carry a natural structure as a piecewise Euclidean simplicial complex.

Example 2.17 *Euclidean buildings of dimension 1 are* metric trees*, i.e. spaces of infinite negative curvature in the sense that all geodesic triangles degenerate to tripods.*

Many interesting examples of locally compact irreducible Euclidean buildings arise from simple algebraic groups over non-Archimedean locally compact fields with a discrete valuation.

Example 2.18 *Let K be a locally compact field with discrete valuation, uniformizer ω, ring of integers $\mathcal{O}$ and residue field k. The Euclidean building attached to $SL(3, K)$ is constructed as follows: It is a simplicial complex built from isometric equilateral Euclidean triangles. The vertices are projective equivalence classes of $\mathcal{O}$-lattices in the K-vector space K^3. Three lattices $\Lambda_0, \Lambda_1, \Lambda_2$ represent the vertices of a triangle if, modulo rescaling and permutation, the inclusion $\omega \cdot \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_0$ holds. $\partial_{Tits}X$ is isomorphic to the spherical building attached to the projective plane over K, and for any vertex $v \in X$ the space of directions $\Sigma_v X$ is isomorphic to the spherical building attached to the projective plane over the residue field k.*

Remark 2.19 (Unsymmetric irreducible rank-2 Euclidean buildings) *There are different locally compact fields with the same residue field, and hence different buildings as in 2.18 with isometric spaces of directions at their vertices. In fact one can construct uncountably many buildings such that the spaces of directions at their vertices are isometric to the spherical building attached to a given projective plane. In this way one can obtain buildings with no non-trivial symmetry and their boundaries are spherical buildings attached to "exotic" topological projective planes.*

2.4 Locally compact topological groups

We will make essential use of a deep result due to Gleason and Yamabe on the approximation of locally compact topological groups by Lie groups:

Theorem 2.20 (cf. [MZ55, p. 153]) *Every locally compact topological group G has an open subgroup G' such that G' can be approximated by Lie groups in the following sense: Every neighborhood of the identity in G' contains an invariant subgroup H such that G'/H is isomorphic to a Lie group.*

Here is a typical **example** of a non-Lie locally compact group: Let T be a locally finite simplicial tree and G its isometry group equipped with the compact-open topology. Vertex stabilizers $Stab(v)$ are open compact subgroups homeomorphic to the Cantor set and can be approximated by finite groups; namely every neighborhood of the identity in $Stab(v)$ contains the stabilizer of a finite set V of vertices, $v \in V \subset T$, as normal subgroup of finite index. Other interesting examples are provided by isometry groups of Euclidean and hyperbolic buildings or more general classes of piecewise Riemannian complexes.

3 Holonomy

Assumption 3.1 *X is a locally compact Hadamard space. $\partial_{Tits}X$ is a thick spherical building of dimension $r-1\geq 1$.*

For a unit sphere $s\subset\partial_{Tits}X$, $0\leq dim(s)<r-1$, we denote by $Link(s)$ the intersection of all closed balls $\bar B_{\pi/2}(\xi)$ centered at points $\xi\in s$. $Link(s)$ is a closed convex subset and consists of the centers of the unit hemispheres $h\subset\partial_{Tits}X$ with boundary s. Note that any two of these hemispheres intersect precisely in s because $\partial_{Tits}X$ is a CAT(1) space. It won't be essential for us but is worth pointing out that $Link(s)$ carries a natural spherical building structure of dimension $dim(Link(s)) = dim(\partial_{Tits}X)-dim(s)-1$, compare Lemma 3.10.1 in [KL96].

For any point $\xi\in s$ we have the natural map

$$Link(s)\to\Sigma_\xi\partial_{Tits}X \tag{11}$$

sending ζ to $\overrightarrow{\xi\zeta}$. Both spaces $Link(s)$ and $\Sigma_\xi\partial_{Tits}X$ inherit a metric and a topology from the Tits metric and cone topology on $\partial_{Tits}X$, and the injective map (11) is a monomorphism in the sense that it preserves both structures, i.e. it is continuous and a Tits isometric embedding[11].

Lemma 3.2 *(11) maps $Link(s)$ onto $Link(\Sigma_\xi s)$.*

If $dim(s)=0$ then $\Sigma_\xi s$ is empty and $Link(\Sigma_\xi s)$ is the full space of directions $\Sigma_\xi\partial_{Tits}X$.

Proof: A direction $\vec v\in Link(\Sigma_\xi s)$ corresponds to a hemisphere $h\subset\Sigma_\xi\partial_{Tits}X$ with boundary $\Sigma_\xi s$. Let $\hat\xi$ be the antipode of ξ in s. Then the union of geodesics of length π with endpoints $\xi,\hat\xi$ and initial directions in h is a hemisphere whose center ζ lies in $Link(s)$ and maps to $\vec v$. $\square$

If $s_1,s_2\subset\partial_{Tits}X$ are unit spheres with $dim(s_1)=dim(s_2)=dim(s_1\cap s_2)\geq 0$ then for any point ξ in the interior of $s_1\cap s_2$ holds $\Sigma_\xi s_1=\Sigma_\xi s_2=\Sigma_\xi(s_1\cap s_2)$ and the identifications

$$Link(s_1)\to Link(\Sigma_\xi(s_1\cap s_2))\leftarrow Link(s_2)$$

yield an *isomorphism*

$$Link(s_1)\leftrightarrow Link(s_2) \tag{12}$$

i.e. a cone topology homeomorphism preserving the Tits metric. The following lemma shows that the identification (12) does not depend on ξ:

Lemma 3.3 *For the points $\zeta_i\in Link(s_i)$ let $h_i\subset\partial_{Tits}X$ be the unit hemispheres with center ζ_i and boundary s_i. Then the points ζ_i correspond to one another under (12) iff the interiors of the hemispheres h_i have non-trivial intersection.*

[11] Recall that the topology induced by the Tits metric is finer than the cone topology.

Proof: If the points ζ_i correspond to one another, i.e. $\overrightarrow{\xi\zeta_1}=\overrightarrow{\xi\zeta_2}$, then the segments $\overline{\xi\zeta_i}$ initially coincide and the interiors of the h_i intersect. Vice versa, if the interiors of the h_i intersect then for any point ξ in the interior of $s_1 \cap s_2$ their intersection $h_1 \cap h_2$ is a neighborhood of ξ in both closed hemispheres $\bar{h}_i$ and therefore $\overrightarrow{\xi\zeta_1}=\overrightarrow{\xi\zeta_2}$. $\square$

We'll now "fill in" the isomorphisms (12) by identifications of convex cores of cross sections of parallel sets in X. This will be acheived by placing different cross sections into the same auxiliary ambient Hadamard space, namely a space of strong asymptote classes, so that their ideal boundaries coincide.

Note that since X has spherical building boundary, 2.2 implies that any apartment $a \subset \partial_{Tits}X$ can be filled by a r-flat $F \subset X$, i.e. $\partial_\infty F = a$. If $s \subset \partial_{Tits}X$ is isometric to a unit sphere then s is contained in an apartment (by [KL96, Proposition 3.9.1]) and hence can be filled by a flat $f \subset X$: $\partial_\infty f = s$. This verifies that the parallel sets defined next are non-empty:

Definition-Description 3.4 *For a unit sphere $s \subset \partial_{Tits}X$ we denote by $P(s) = P^X(s)$ the union of all flats with ideal boundary s. $P(s)$ is a non-empty convex subset and splits metrically as*

$$P(s) \cong \mathbb{R}^{1+\dim s} \times CS(s). \tag{13}$$

The subsets $\mathbb{R}^{1+\dim s} \times \{point\}$ are the flats with ideal boundary s. $CS(s)$ is again a locally compact Hadamard space which we call the **cross section** *of $P(s)$. For any flat $f \subset X$, $P(f) := P(\partial_{Tits}f)$ denotes its* **parallel set***, i.e. the union of all flats parallel to f, and $CS(f) := CS(\partial_{Tits}f)$ denotes the cross section.*

Observe that $\partial_{Tits}CS(s) = Link(s)$. Namely a ray in $CS(s)$ determines a flat half space in X whose ideal boundary is a hemisphere h in $\partial_{Tits}X$ with $\partial h = s$; vice versa, any such hemisphere in $\partial_{Tits}X$ can be filled by a half-flat in X. For any point $\xi \in s$ the natural map $CS(s) \to X_\xi$ assigning to a point x the ray $\overline{x\xi}$ is an isometric embedding because for $x_1, x_2 \in CS(s)$ the triangle with vertices x_1, x_2, ξ has right angles at the x_i.

Lemma 3.5 *Let $s_1, s_2 \subset \partial_{Tits}X$ be unit spheres with $dim(s_1) = dim(s_2) = dim(s_1 \cap s_2) \geq 0$. If ξ is an interior point of $s_1 \cap s_2$ then the images of the isometric embeddings*

$$CS(s_i) \hookrightarrow X_\xi \tag{14}$$

have the same ideal boundary. Furthermore the resulting identification of ideal boundaries coincides with the earlier identification (12).

Proof: Let $\zeta_i \in \partial_{Tits}CS(s_i) = Link(s_i)$ be points corresponding to each other under (12), i.e. $\overrightarrow{\xi\zeta_1}=\overrightarrow{\xi\zeta_2}$. The segments $\overline{\xi\zeta_i}$ initially coincide, i.e. they share a non-degenerate segment $\overline{\xi\eta}$. Let r_i be a ray in $CS(s_i)$ asymptotic to ζ_i and $r_i' \subset CS(s_i)$ be the ray with same initial point but asymptotic to η. Then r_i and r_i' have the same image in X_ξ under (14) because they lie in a flat half-plane whose boundary geodesic is asymptotic to ξ. Since the rays r_1' and r_2' are asymptotic this shows that the images of r_1 and r_2 in X_ξ are asymptotic rays. $\square$

The Tits boundaries $\partial_{Tits}CS(s) = Link(s)$ contain top-dimensional unit spheres and 2.2 implies that the cross sections $CS(s)$ have a convex core.

Lemma 3.6 *If $s \subset \partial_{Tits} X$ is a singular sphere then $Link(s)$ does not splitt off a spherical join factor. As a consequence, the convex core of $CS(s)$ has no Euclidean factor.*

Proof: If $Link(s)$ would have a spherical join factor then this factor would be contained in all maximal unit spheres in $Link(s)$. Hence the intersection of all apartments $a \subset \partial_{Tits} X$ with $a \supset s$ would contain a larger sphere than s. This is impossible because $\partial_{Tits} X$ is a thick spherical building and the singular sphere s is therefore an intersection of apartments. $\square$

Fix a simplex $\tau \subset \partial_{Tits} X$ and choose a point ξ in the interior of τ. Then the cross sections $CS(s)$ for all singular spheres $s \supset \tau$ with $dim(s) = dim(\tau)$ isometrically embed into the same ambient Hadamard space X_ξ. By 3.5 their images have equal ideal boundaries and the boundary identification is given by (12). According to the proof of part 2 of 2.2, the convex cores of the $CS(s)$ are mapped to parallel layers of a flat strip and their boundary identifications (12) can be induced by isometries which are unique in view of 2.7 and 3.6. In this way we can compatibly identify the convex cores in consideration to a Hadamard space C_τ and there is a canonical isomorphism

$$\partial_{Tits} C_\tau \stackrel{\cong}{\longrightarrow} \Sigma_\tau \partial_{Tits} X. \tag{15}$$

If σ, τ are top-dimensional simplices in the same singular sphere $s \subset \partial_{Tits} X$ then there is a canonical perspectivity isometry

$$persp_{\sigma\tau} : C_\sigma \leftrightarrow C_\tau : persp_{\tau\sigma} \tag{16}$$

because both sets are identified with the convex core of $CS(s)$. The map of ideal boundaries induced by (16) turns via (15) into an isomorphism

$$\Sigma_\sigma \partial_{Tits} X \leftrightarrow \Sigma_\tau \partial_{Tits} X \tag{17}$$

(of topological buildings) which can be described inside the Tits boundary as follows: $\vec{u} \in \Sigma_\sigma \partial_{Tits} X$ and $\vec{v} \in \Sigma_\tau \partial_{Tits} X$ correspond to each other if they are tangent to the same hemisphere in $\partial_{Tits} X$ with boundary s. (17) is independent of the choice of $s \supset \sigma \cup \tau$.

Let $\tau, \tilde{\tau} \subset \partial_{Tits} X$ be simplices of equal dimension and suppose that they are *projectively equivalent*, i.e. there exists a sequence $\tau = \tau_0, \ldots, \tau_m = \tilde{\tau}$ of simplices of the same dimension so that any two successive simplices τ_i, τ_{i+1} are top-dimensional simplices in a singular sphere. By composing the natural isometries (16), $C_{\tau_i} \to C_{\tau_{i+1}}$, we obtain an isometry

$$C_\tau \to C_{\tilde{\tau}} \tag{18}$$

Definition 3.7 *The topological space*

$$Hol^X(\tau, \tilde{\tau}) \subseteq Isom(C_\tau, C_{\tilde{\tau}})$$

is defined as the closure of the subset of isometries (18). The **holonomy group**

$$Hol(\tau) = Hol^X(\tau) \subseteq Isom(C_\tau)$$

at the simplex τ is defined as the topological group $Hol^X(\tau, \tau)$.

For a face $\tau \subset \partial_{Tits} X$ we'd now like to relate the holonomy groupoid on the space C_τ to the holonomy groupoid on X. This will be useful in the proof of 3.8 because it allows to reduce the study of the holonomy action to the rank 2 case.

Let $s, S \subset \partial_{Tits} X$ be unit spheres so that $s \subset S$. Let us denote by $s^\perp \subset S$ the subsphere complementary to s, i.e. $s^\perp = Link_S(s)$ and $S = s \circ s^\perp$. There are natural inclusions $Link(S) \subset Link(s)$ and $P(S) \subset P(s)$. More precisely holds

$$Link(S) \cong Link_{Link(s)}(s^\perp) \tag{19}$$

and

$$CS(S) \cong CS^{CS(s)}(s^\perp). \tag{20}$$

Assume now that the spheres s, S are singular and that $\tau \subset s$ and $\mathcal{T} \subset S$ are top-dimensional simplices in these spheres so that τ is a face of $\mathcal{T}$. The identification $Link(s) \cong \Sigma_\tau \partial_{Tits} X$ carries $s^\perp$ to $\Sigma_\tau S$ and $Link_{Link(s)}(s^\perp)$ to $Link(\Sigma_\tau S)$. $core(CS(s)) \cong C_\tau$ carries $core(CS(S)) \cong C_{\mathcal{T}}$ to $core(CS^{C_\tau}(\Sigma_\tau S)) \cong C^{C_\tau}_{\Sigma_\tau \mathcal{T}}$ and hence induces a canonical identification

$$C_{\mathcal{T}} \xrightarrow{\cong} C^{C_\tau}_{\Sigma_\tau \mathcal{T}}. \tag{21}$$

Two faces $\mathcal{T}_1, \mathcal{T}_2 \supset \tau$ are top-dimensional simplices in the same singular sphere S iff the $\Sigma_\tau \mathcal{T}_i$ are top-dimensional simplices in the same singular sphere in $\Sigma_\tau \partial_{Tits} X$. Let us assume that this were the case. Then the perspectivity $C_{\mathcal{T}_1} \leftrightarrow C_{\mathcal{T}_2}$ induces the perspectivity $C^{C_\tau}_{\Sigma_\tau \mathcal{T}_1} \leftrightarrow C^{C_\tau}_{\Sigma_\tau \mathcal{T}_2}$. We obtain an embedding

$$Hol^{C_\tau}(\Sigma_\tau \mathcal{T}) \hookrightarrow Hol^X(\mathcal{T}). \tag{22}$$

We come to the main result of this section, namely that in the irreducible case the holonomy groups are non-trivial, even large:

Proposition 3.8 *Suppose that, in addition to 3.1, the spherical building $\partial_{Tits} X$ is irreducible of dimension ≥ 1. Then for any panel $\tau \subset \partial_{Tits} X$ and any $\eta \in \partial_\infty C_\tau$, the parabolic stabilizer P_η in $Hol(\tau)$ acts transitively on $\partial_\infty C_\tau \setminus \{\eta\}$.*

Proof: Let us first consider the case $\dim \partial_{Tits} X = 1$. The panel τ is then a vertex ξ. The action of $Hol(\xi)$ at infinity on $\partial_\infty C_\xi \cong \Sigma_\xi \partial_{Tits} X$ can be analysed inside $\partial_{Tits} X$:

Sublemma 3.9 *For every vertex ξ, $Hol(\xi)$ acts 2-fold transitively on $\Sigma_\xi \partial_{Tits} X$.*

Proof: Denote by l the length of Weyl chambers. Irreducibility implies $l \leq \pi/3$. Consider two vertices ξ_1 and ξ_2 of distance $2l$ and let μ be the midpoint of $\overline{\xi_1 \xi_2}$. Extend $\overline{\xi_1 \mu \xi_2}$ in an arbitrary way to a (not necessarily globally minimizing) geodesic $\overline{\eta_1 \xi_1 \mu \xi_2 \eta_2}$ of length $4l$. By irreducibility, this geodesic is contained in an apartment α for any choice of η_1 and η_2. Denote by $\hat{\mu}$ the antipode of μ in α and let $\zeta \notin \alpha$ be some neighboring vertex of μ. Then $\angle_{Tits}(\zeta, \xi_i) = \pi$ and we can form the composition of natural maps (17):

$$\Sigma_{\xi_1} \partial_{Tits} X \to \Sigma_\zeta \partial_{Tits} X \to \Sigma_{\xi_2} \partial_{Tits} X.$$

Varying η_1, η_2, ζ we get plenty of maps $\Sigma_{\xi_1}\partial_{Tits}X \to \Sigma_{\xi_2}\partial_{Tits}X$ sending $\overrightarrow{\xi_1\xi_2}=\overrightarrow{\xi_1\mu}$ to $\overrightarrow{\xi_2\xi_1}=\overrightarrow{\xi_2\mu}$ and $\overrightarrow{\xi_1\eta_1}$ to $\overrightarrow{\xi_2\eta_2}$. We can compose these and their inverses to obtain selfmaps of $\Sigma_{\xi_1}\partial_{Tits}X$ and see that the stabilizer of $\overrightarrow{\xi_1\xi_2}$ in $Hol(\xi_1)$ acts transitively on the complement of $\overrightarrow{\xi_1\xi_2}$. Since $\partial_{Tits}X$ is thick, $\Sigma_\xi\partial_{Tits}X$ contains at least three points and it follows that $Hol(\xi)$ acts 2-fold transitively. $\square$

Proof of 3.8 continued: If P_η does not act transitively on $\partial_\infty C_\tau \setminus \{\eta\}$ then, by 3.9, there is a non-trivial axial isometry $\alpha \in Hol(\xi)$ (fixing η), and for any $\zeta \in \partial_\infty C_\tau \setminus \{\eta\}$ there is a conjugate α_ζ of α with attractive fixed point η and repulsive fixed point ζ. For $\zeta_1, \zeta_2 \neq \eta$ the isometries $\alpha_{\zeta_2}^{-n} \circ \alpha_{\zeta_1}^n \in P_\eta$ subconverge to $\beta \in P_\eta$ with $\beta\zeta_1 = \zeta_2$. This concludes the proof in the 1-dimensional case.

The general case $\dim \partial_{Tits}X \geq 1$ can be derived: Thanks to irreducibility, we can find for every panel τ an adjacent panel $\hat\tau$ so that $\mu := \tau \cap \hat\tau$ has codimension 2 and $\angle_\mu(\tau, \hat\tau) < \pi/2$. The building $\partial_{Tits}C_\mu \cong \Sigma_\mu\partial_{Tits}X$ is 1-dimensional irreducible. $\Sigma_\mu\tau$ is a vertex and $Hol^{C_\mu}(\Sigma_\mu\tau)$ acts by isometries on $C^{C_\mu}_{\Sigma_\mu\tau} \cong C_\tau$. We get an embedding $Hol^{C_\mu}(\Sigma_\mu\tau) \hookrightarrow Hol^X(\tau)$ as in (22). Our result in the 1-dimensional case implies the assertion. $\square$

Example 3.10 *If $\partial_{Tits}X$ is the spherical building associated to a projective plane (with more than three points) then $Hol(\xi)$ acts 3-fold transitive on $\Sigma_\xi\partial_{Tits}X$ (by "Möbius transformations").*

4 Rank one: Rigidity of highly symmetric visibility spaces

Assumption 4.1 *Let Y be a locally compact Hadamard space with at least three ideal boundary points, with extendible rays, and which is minimal in the sense that $Y = core(Y)$. Suppose furthermore that $H \subseteq Isom(Y)$ is a closed subgroup so that for each ideal boundary point $\xi \in \partial_\infty Y$ the parabolic stabilizer P_ξ in H acts transitively on $\partial_\infty Y \setminus \{\xi\}$.*

In particular, Y has the visibility property. For any complete geodesic c we denote by $P(c)$ the *parallel set* of c, that is, the union of all geodesics parallel to c. It splits as $c \times cpt$ and contains a distinguished *central* geodesic. By 2.14 there exist axial elements in H, and hence the stabilizer of any central geodesic contains axial elements. In particular, H acts cocompactly on Y and Y is large-scale hyperbolic (in the sense of Gromov). For any oriented central geodesic c there is a canonical homomorphism

$$trans : Stab(c) \to \mathbb{R} \tag{23}$$

given by the translational part. Its image is non-trivial closed, so either infinite cyclic or $\mathbb{R}$. The main result of this section is:

Theorem 4.2 *1. (23) is surjective iff complete geodesics in Y do not branch. In this case, H is a simple Lie group, there exists a negatively curved symmetric space Y_{model} and a homeomorphism*

$$\beta : \partial_\infty Y \xrightarrow{\cong} \partial_\infty Y_{model}$$

which carries H_o to $Isom_o(Y_{model})$: $\beta H \beta^{-1} = Isom_o(Y_{model}) \subset Homeo(\partial_\infty Y_{model})$.
2. The image of (23) is cyclic iff Y splits metrically as tree $\times$ cpt.
3. If T_1, T_2 are two geodesically complete locally compact metric trees (with at least three ideal boundary points), and if there are embeddings of topological groups $H \hookrightarrow Isom(T_i)$ satisfying 4.1, then there is an H-equivariant homothety $T_1 \to T_2$.

4.2 is a combination of the results 4.24, 4.15 and 4.20.

4.1 General properties

Lemma 4.3 *Let $\rho : [0, \infty) \to Y$ be a ray asymptotic to the geodesic c. Then ρ is strongly asymptotic to $P(c)$, i.e. $d(\rho(t), P(c)) \to 0$.*

Proof: Assume that ρ has strictly positive distance d from $P(c)$. The stabilizer of $P(c)$ contains axial elements with repulsive fixed point $\rho(\infty)$. Applying them to ρ we can construct a geodesic at positive distance from $P(c)$, contradicting the definition of parallel set. $\square$

Lemma 4.4 *Let c be a geodesic and $B_\pm$ Busemann functions centered at the ideal endpoints $c(\pm\infty)$. Then the set where the 2-Lipschitz function $B_+ + B_-$ attains its minimum is precisely $P(c)$.*

Proof: Clear. $\square$

Lemma 4.5 *For every $h > 0$ there exists $\alpha = \alpha(h) < \pi$ so that the following implication holds: If $c : \mathbb{R} \to Y$ is a geodesic, y a point with $\angle_y(c(-\infty), c(+\infty)) \geq \alpha$ then $d(y, P(c)) \leq h$.*

Proof: Suppose that for some positive h there is no $\alpha < \pi$ with this property. Then there exist points y_n of distance $\geq h$ from $P(c)$ so that $\alpha_n = \angle_{y_n}(c(-\infty, +\infty)) \to \pi$. (All central geodesics are equivalent modulo the action of H.) This implies that there exist points y'_n (on $\overline{y_n \pi_{P(c)}(y_n)}$) so that $d(y'_n, P(c)) = h$ and $\angle_{y'_n}(c(\pm\infty), \pi_{P(c)}(y_n)) \to \pi/2$[12]. Since H acts cocompactly, we may assume that the y'_n subconverge. Taking a limit, we can construct a geodesic parallel to c and at positive distance h from $P(c)$, a contradiction. $\square$

4.2 Butterfly construction of small axial isometries

Consider two rays $\rho_i : [0, \infty) \to Y$ emanating from the same point y and assume that $\angle_y(\rho_1, \rho_2) < \pi$. Let $c_i : \mathbb{R} \to Y$ be extensions of the rays ρ_i to complete geodesics. We produce an isometry ψ preserving the parallel set $P(c_1)$ by composing four parabolic isometries: Let $p_{i,\pm} \in P(c_i(\pm\infty))$ be the isometry which moves $c_i(\mp\infty)$ to $c_{3-i}(\mp\infty)$. Then

$$\psi := p_{1,+}^{-1} p_{2,-} p_{2,+}^{-1} p_{1,-}$$

[12] For a closed convex subset C of a Hadamard space X, $\pi_C : X \to C$ denotes the closest point projection.

preserves $P(c_1)$ and translates it by the displacement

$$\delta_\psi = \left(\sum B_{i,\pm}(y)\right) - \min(B_{1,+} + B_{2,-}) - \min(B_{1,-} + B_{2,+}) \geq 0$$

towards $c_1(+\infty)$. The displacement δ_ψ is positive and ψ axial iff one of the angles $\angle_y(c_1(\pm\infty), c_2(\mp\infty))$ is smaller than π. On the other hand, δ_ψ is bounded from above by twice the sum of the distances from y to the parallel sets $Y(c_1(\pm\infty), c_2(\mp\infty))$.

Lemma 4.6 *If $\angle_y(\rho_1, \rho_2) \leq \pi - \alpha(h)$ then $\delta_\psi \leq 4h$.*

Proof: Since $\angle_y(c_1(\pm\infty), c_2(\mp\infty)) \geq \alpha(h)$, 4.5 implies $d(y, P(c_1(\pm\infty), c_2(\mp\infty))) \leq h$. Hence $B_{1,\pm}(y) + B_{2,\mp}(y) - \min(B_{1,\pm} + B_{2,\mp}) \leq 2h$ and the claim follows. $\square$

4.3 The discrete case

Assumption 4.7 *(23) has cyclic image: The stabilizer in H of any central geodesic has a discrete orbit on the central geodesic.*

Then there is a positive lower bound for the displacement of axial isometries in H. By 4.6 there exists $\alpha_0 > 0$ such that: If the rays ρ_1 and ρ_2 initiate in the same point y and have angle $\angle_y(\rho_1, \rho_2) < \alpha_0$ then $\rho_i(\infty)$ have the same y-antipodes (i.e. for a third ray initiating in y we have $\angle_y(\rho, \rho_1) = \pi$ iff $\angle_y(\rho, \rho_2) = \pi$).

Lemma 4.8 (No small angles between rays) *If the rays ρ_1 and ρ_2 initiate in the same point y and have angle $< \alpha_0$ then they initially coincide, i.e. $\rho_1(t) = \rho_2(t)$ for small positive t.*

Proof: For small positive t holds $\angle_{\rho_1(t)}(\rho_1(\infty), \rho_2(\infty)) < \alpha_0$, so $\rho_i(\infty)$ have the same $\rho(t)$-antipodes[13] and $\rho_2(t) = \rho_1(t)$. $\square$

Lemma 4.9 (Bounded Diving Time) *If $\rho : [0, \infty) \to Y$ is a ray asymptotic to c and if $d(\rho(0), P(c)) \leq h$ then $\rho(t) \in P(c)$ for all $t \geq h/\sin(\alpha_0)$.*

Proof: We extend ρ to a geodesic c'. ρ is strongly asymptotic to $P(c)$ (4.3). Hence there exist $y_n \in P(c)$ tending to $\rho(\infty)$ so that the rays $\rho_n = \overline{y_n c'(-\infty)}$ Hausdorff converge to c'. $\angle_{y_n}(c(-\infty), c'(-\infty)) \to 0$ and ρ_n therefore initially lies in $P(c)$ for large n (4.8). Outside $P(c)$ the derivative of $d(\rho_n(t), P(c))$ is $\leq -\sin(\alpha_0)$ whence the estimate. $\square$

Corollary 4.10 (Discrete Branching) *There exist branching complete geodesics: Any two strongly asymptotic geodesics share a ray. Furthermore, the set of branching points on any geodesic c is discrete.*

[13] Let x be a point in the Hadamard space X. Then $\xi, \eta \in \partial_\infty X$ are x-*antipodal* to each other if there exists a geodesic passing through x and asymptotic to ξ, η.

Proof: The first assertion is clear from 4.9. The second follows from local compactness: Let c_n be a sequence of geodesics so that $c_n \cap c = c_n((-\infty, 0])$ and the branching points $c_n(0)$ are pairwise distinct and converge. Then, for large n, the points $c_n(1)$ are uniformly separated (by 4.8) but they form a bounded subset, contradiction. $\square$

Proposition 4.11 (Local Conicality) *Let $\rho : \mathbb{R}^+ \to Y$ be a geodesic ray, $\sigma : [0, l] \to Y$ a segment so that $\rho(0) = \sigma(0)$. Then there exists $t_0 > 0$ so that the triangle with vertices $\sigma(0), \sigma(t_0), \rho(\infty)$ spans a flat half-strip and is contained in a flat strip.*

Proof: Denote by $\rho_t : \mathbb{R}^+ \to Y$ the ray emanating from $\sigma(t)$ and asymptotic to ρ. ρ_t can be extended to a geodesic c_t and there is a parallel geodesic c'_t strongly asymptotic to ρ. The branch point of c'_t and ρ tends to $\rho(0)$ as $t \to 0$. Discreteness of branching points on geodesics (and hence rays) implies that $c'(t)$ passes through $\rho(0)$ for small t, and $\sigma\big|_{[0,t]}$ lies in the flat strip bounded by c_t and c'_t. $\square$

Consequence 4.12 *Let $\rho_1, \rho_2 : \mathbb{R}^+ \to Y$ be rays emanating from the same point y and with angle $\angle_y(\rho_1, \rho_2) = \alpha$. Then ρ_1 can be extended to a complete geodesic c_1 such that $\angle_y(\rho_2(\infty), c_1(-\infty)) = \pi - \alpha$.*

Consequence 4.13 (Fattening half-strips) *Let $\eta \in \partial_\infty Y$ and suppose that $\sigma : [0, b] \to Y$, $0 < b$, is a segment which is contained in a complete geodesic (ray). Assume that the ideal triangle $\Delta(\sigma(0), \sigma(b), \eta)$ bounds a flat half-strip. Then we can extend the segment σ to a longer segment $\sigma : [a, b] \to Y$, $a < 0$, so that the ideal triangle $\Delta(\sigma(a), \sigma(b), \eta)$ bounds a flat half-strip.*

Proof: We assume $0 < \angle_{\sigma(0)}(\sigma(b), \eta) < \pi$ because otherwise the claim holds trivially. Let $\rho : \mathbb{R}^+ \to Y$ be a ray extending σ, i.e. $\rho\big|_{[0,b]} \equiv \sigma$. By 4.12, we can find a geodesic c extending ρ and a flat strip S bounded by c so that the ray $\overline{\sigma(0)\eta}$ is initially contained in S. Then $\angle_{\sigma(0)}(c(-\infty), \eta) + \angle_{\sigma(0)}(c(+\infty), \eta) = \pi$. For $a < 0$ sufficiently close to 0 the ideal triangle $\Delta(c(a), c(0) = \sigma(0), \eta)$ bounds a flat half-strip, hence $\angle_{c(a)}(c(b), \eta) + \angle_{c(b)}(c(a), \eta) = \pi$ and $\Delta(c(a), c(b), \eta)$ bounds a flat half-strip. $\square$

Corollary 4.14 *The angle between any two rays emanating from the same point is 0 or π.*

Proof: Suppose that $\rho_1, \rho_2 : \mathbb{R}^+ \to Y$ are two rays emanating from the same point y with angle $\angle_y(\rho_1, \rho_2) = \alpha$. For small t, the ideal triangle $\Delta(\rho_1(0), \rho_1(t), \rho_2(\infty))$ bounds a flat half-strip (4.11). By 4.13 and local compactness we can extend ρ_1 to a complete geodesic $c_1 : \mathbb{R} \to Y$ so that $\angle_{c_1(-t)}(\rho_1(\infty), \rho_2(\infty)) = \alpha$ for all $-t \leq 0$. Since Y is large-scale hyperbolic this implies that $\alpha = 0$ or π. $\square$

Proposition 4.15 *Y splits as tree $\times$ compact.*

Proof: According to 4.14, for every $y \in Y$ the union Sun_y of all rays initiating in y is a minimal closed convex subset isometric to a metric tree. 2.2 implies the assertion. $\square$

Proposition 4.16 *Let Y' be a locally compact Hadamard space with extendible rays and suppose that $T = core(Y')$ exists and is a metric tree. Then $Y' \cong T \times cpt$.*

Proof: The tree T is locally compact and geodesically complete, so it is also discrete.

Sublemma 4.17 *The nearest point projection $\pi_T : Y' \to T$ restricts to an isometry on every ray r in Y'.*

Proof: We can extend r to a complete geodesic l and observe that the distance $d(\cdot, T)$ from T is constant on l because $l(\pm\infty) \in \partial_\infty T$. It follows that π_T restricts on l to an isometry. $\square$

Sublemma 4.18 *Let $y \in Y'$ and $\xi_1, \xi_2 \in \partial_\infty Y'$ so that $\angle_{\pi_T y}(\xi_1, \xi_2) = \pi$. Then $\angle_y(\xi_1, \xi_2) = \pi$.*

Proof: For points y_i on the rays $\overline{y\xi_i}$ we have

$$d(y_1, y_2) \geq d(\pi_T y_1, \pi_T y_2) = d(\pi_T y_1, \pi_T y) + d(\pi_T y, \pi_T y_2)$$
$$= d(y_1, y) + d(y, y_2) \geq d(y_1, y_2).$$

Thus equality holds and $\angle_y(y_1, y_2) = \pi$. $\square$

Sublemma 4.19 *Let $\xi \in \partial_\infty Y'$ and c be a geodesic in Y' not asymptotic to ξ. Then there is a point $y \in c$ with $\angle_y(l(\pm\infty), \xi) = \pi$.*

Proof: Let y be the point which projects via π_T to the center of the tripod in T spanned by the ideal points $l(\pm\infty), \xi$ and apply 4.18. $\square$

Thus any two rays in Y' with same initial point have angle 0 or π and 4.16 follows. $\square$

4.3.1 Equivariant rigidity for trees

Suppose that T_1 and T_2 are geodesically complete locally compact metric trees with at least three boundary points, that the locally compact topological group H is embedded into their isometry groups, $H \subseteq Isom(T_i)$, and that the induced boundary actions of H on $\partial_\infty T_i$ satisfy 4.1.

Proposition 4.20 *Every H-equivariant homeomorphism $\partial_\infty T_1 \to \partial_\infty T_2$ is induced by an H-equivariant homothety $T_1 \to T_2$.*

Proof: Maximal compact subgroups $K \subset H$ whose fixed point set on T_i is a vertex (and not the midpoint of an edge) can be recognized from their dynamics at infinity: There exist three ideal boundary points so that one can map anyone to any other of them by isometries in K while fixing the third. Adjacency of vertices can be characterised in terms of stabilizers: The vertices $v, v' \in T_i$ are adjacent iff $Stab(v) \cap Stab(v')$ is contained in precisely two maximal compact vertex stabilizers. It follows that there is a H-equivariant combinatorial isomorphism $T_1 \to T_2$. It is a homothety because all edges in T_i have equal length. $\square$

4.4 The non-discrete case

Assumption 4.21 *(23) is surjective: The stabilizer in H of any central geodesic c acts transitively on c.*

Lemma 4.22 *Let G be an open subgroup of H and c a central geodesic. Then $Stab_G(c)$ acts transitively on c.*

Proof: We choose elements $h_n \in Stab_H(c)$ with $trans(h_n) = 1/n$. They form a bounded sequence and subconverge to an elliptic element $k \in Fix_H(c)$. Then $(k^{-1}h_n)$ subconverges to e and there exist arbitrarily large $m \neq n$ so that $h_m^{-1}h_n$ is axial and contained in G. This shows that $Stab_G(c)$ contains axial elements with arbitrarily small non-vanishing translational part. □

Consequence 4.23 *Any open subgroup of H acts cocompactly on Y.*

Proposition 4.24 *There exist a negatively curved symmetric space Y_{model}, an isomorphism $H_o \overset{\cong}{\to} Isom_o(Y_{model})$ and an equivariant homeomorphism $\partial_\infty Y \to \partial_\infty Y_{model}$.*

Proof: Suppose that $G' \subseteq H$ is an open subgroup and that K is an invariant compact subgroup of G'. 4.22 shows that the G'-invariant non-empty closed convex subset $Fix(K)$ has full boundary at infinity: $\partial_\infty Fix(K) = \partial_\infty Y$. The minimality of Y implies $Fix(K) = Y$ and $K = \{e\}$. Applying 2.20 we conclude that H is a Lie group.

Sublemma 4.25 *H has no non-trivial invariant abelian subgroup A.*

Proof: A would have a non-empty fixed point set in the geometric compactification $\overline{Y}$. If A fixes points in Y itself then $Fix(A) = Y$ and $A = \{e\}$ by the cocompactness of H and the minimality of Y. If all fixed points of A lie at infinity then there are at most two. This leads to a contradiction because the fixed point set of A on $\partial_\infty Y$ is H-invariant, hence full or empty. □

So H is a semisimple Lie group with trivial center and $H_o \cong Isom_o(Y_{model})$ for a symmetric space Y_{model} of noncompact type and without Euclidean factor.

Sublemma 4.26 *Y_{model} has rank one.*

Proof: If $rank(Y_{model}) \geq 2$ then the subgroup of translations along a maximal flat in Y_{model} acts on Y as a parabolic subgroup (because no subgroup $\cong \mathbb{R}^2$ in $Isom(Y)$ can contain axial isometries) and fixes exactly one point on $\partial_\infty Y$. Maximal flats in Y_{model} containing parallel singular geodesics yield the same fixed point in $\partial_\infty Y$ and it follows that H_o would have a fixed point on $\partial_\infty Y$, contradiction. □

It remains to construct the equivariant homeomorphism of boundaries. Axial isometries in $Isom(Y)$ have the property that their conjugacy class never accumulates at the identity. Therefore if $h \in H_o$ acts as a pure parabolic (see definition 2.10) on Y_{model} then it acts as a parabolic on Y. Hence the stabilizer of $\xi_0 \in \partial_\infty Y_{model}$ in H_o fixes a unique point $\xi \in \partial_\infty Y$ and we obtain an H_o-equivariant, and hence continuous surjective map $\partial_\infty Y_{model} \to \partial_\infty Y$. It must be injective, too, because any two stabilizers of distinct points in $\partial_\infty Y_{model}$ generate H_o but H_o has no fixed point on $\partial_\infty Y$. This concludes the proof of 4.24. □

Proposition 4.27 *Complete geodesics in Y don't branch.*

Proof: $h \in H_o$ acts as a pure parabolic on Y iff it does so on Y_{model}. The purely parabolic stabilizer $N_\xi \subset H_o$ of $\xi \in \partial_\infty Y$ is a simply connected nilpotent Lie group and acts simply transitively on $\partial_\infty Y \setminus \{\xi\}$. Let $\tau \in H_o$ be any axial isometry acting on Y with attractive fixed point ξ. Then

$$\lim_{n\to\infty} \tau^{-n}\phi\tau^n = e. \tag{24}$$

for all $\phi \in N_\xi$. Let c be a geodesic in Y asymptotic to both fixed points of τ at infinity and let $\phi \in N_\xi$ be non-trivial. τ acts as an isometry on the compact cross section of $P(c)$ and we can choose a sequence $n_k \to \infty$ so that $d(c, \tau^{n_k}c) \to 0$.

$$d(\phi\tau^{n_k}c(0), \tau^{n_k}c(0)) = d(\tau^{-n_k}\phi\tau^{n_k}c(0), c(0)) \to 0$$

implies that ϕc is strongly asymptotic to c. These two geodesics can't intersect because ϕ is not elliptic. (N_ξ has no non-trivial elliptic elements!) The argument shows that distinct stronlgy asymptotic geodesics are disjoint and hence geodesics in Y don't branch. $\Box$

Proof of 1.7: 1.7 is not much more than a reformulation of 4.2. As in the proof of 3.8 we deduce from the 2-fold transitivity of the action of $Isom(C)$ on $\partial_\infty C$ that the parabolic stabilizer of any $\eta \in \partial_\infty C$ acts transitively on $\partial_\infty C \setminus \{\eta\}$. Then C and $Isom(C)$ satisfy assumption 4.1 and assertion follows from 4.2 and 4.16. $\Box$

5 Geodesically complete Hadamard spaces with building boundary

5.1 Basic properties of parallel sets

Assumption 5.1 *X is a locally compact Hadamard space with extendible rays and $\partial_{Tits}X$ is a spherical building of dimension $r-1 \geq 1$.*

Lemma 5.2 *Every flat half-plane h in X is contained in a flat plane.*

Proof: Let c be the boundary geodesic of the flat half-plane h and denote $\xi_\pm := c(\pm\infty)$. Let $\eta \in \partial_\infty h$ be so close to ξ_+ that the arc $\overline{\eta\xi_+}$ in $\partial_{Tits}X$ is contained in a closed chamber, and extend the ray $\overline{\eta c(0)}$ to a geodesic c'. c' bounds a flat half-plane h' which contains ξ_+ in its ideal boundary. The canonical isometric embedding $CS(\{\xi_+, \xi_-\}) \hookrightarrow X_{\xi_+}$ sends h to a ray and h' to a geodesic extending this ray. This implies that h is contained in a flat plane. $\Box$

Corollary 5.3 *For any flat $f \subset X$ the cross section $CS(f)$ is again a locally compact Hadamard space with extendible rays, and $\partial_{Tits}CS(f)$ is a spherical building of dimension* $\dim(\partial_{Tits}X) - \dim(f)$.

Proof: 5.2 implies that for any geodesic l the cross section $CS(l)$ has extendible rays. Now we proceed by induction on the dimension of f using

$$CS^X(f) \cong CS^{CS^X(f')}(CS^f(f'))$$

for flats $f' \subset f$. $\square$

Corollary 5.4 *Every flat is contained in a r-flat.*

Proposition 5.5 *Suppose the geodesics $c_1, c_2 \subset X$ have a ray ρ in common. Then there are two maximal flats whose intersection is a halfapartment.*

Proof: Denote $\xi := \rho(\infty) = c_i(\infty)$ and $\xi_i := c_i(-\infty)$. There exist geodesics γ_i of length π in $\partial_{Tits}X$ joining ξ and ξ_i so that their intersection $\gamma_1 \cap \gamma_2$ is a non-degenerate arc $\overline{\xi\eta}$. The geodesics c_i project to geodesics $\bar{c}_i$ in the space of strong asymptote classes X_η, and for any $\rho(0)$-antipode $\hat{\eta}$ of η the geodesics $\bar{c}_i$ are in fact contained in the projection to X_η of the cross section $CS(\{\eta, \hat{\eta}\})$. The geodesics $\bar{c}_i$ share a ray but do not coincide because they have different ideal endpoints $\bar{c}_i(-\infty) = \overrightarrow{\eta\xi_i} \in \Sigma_\eta\partial_{Tits}X \cong \partial_\infty CS(\{\eta, \hat{\eta}\})$. We may proceed by induction on the dimension of the Tits boundary of the cross section until we find a flat f so that $CS(f)$ has discrete Tits boundary and contains two geodesics whose intersection is a ray. These geodesics correspond to maximal flats in $P(f)$ with the desired property. $\square$

Reformulation 5.6 *If there are branching geodesics in X then there exists a flat $f \subset X$ so that $\partial_{Tits}f$ is a wall in $\partial_{Tits}X$ and $CS(f)$ contains branching geodesics.*

5.2 Boundary isomorphisms

Definition 5.7 *Let X' be another space satisfying 5.1. A* **boundary isomorphism** *is a cone topology homeomorphism*

$$\phi : \partial_\infty X \longrightarrow \partial_\infty X' \tag{25}$$

which at the same time is a Tits isometry, i.e. it is an isomorphism of topological spherical buildings, cf. [BS87]. We denote by $Iso(\partial_\infty X, \partial_\infty X')$ the space of all boundary isomorphisms $\partial_\infty X \to \partial_\infty X'$ equipped with the compact-open topology, and by $Aut(\partial_\infty X)$ the topological group $Iso(\partial_\infty X, \partial_\infty X)$.

A boundary isomorphism (25) induces for all simplices $\tau \subset \partial_{Tits}X$ an isomorphism of topological buildings

$$\Sigma_\tau\partial_{Tits}X \longrightarrow \Sigma_{\phi\tau}\partial_{Tits}X'. \tag{26}$$

The induced homeomorphisms

$$Iso(\Sigma_\tau\partial_{Tits}X, \Sigma_{\tilde{\tau}}\partial_{Tits}X) \to Iso(\Sigma_{\phi\tau}\partial_{Tits}X, \Sigma_{\phi\tilde{\tau}}\partial_{Tits}X)$$

carry $Hol^X(\tau, \tilde{\tau})$ to $Hol^{X'}(\phi\tau, \phi\tilde{\tau})$ and thereby induce isomorphisms of topological groups

$$Hol^X(\tau) \longrightarrow Hol^{X'}(\phi\tau). \tag{27}$$

Assumption 5.8 *In addition to 5.1 the building $\partial_{Tits}X$ is thick and irreducible.*

According to 5.3, C_τ has extendible rays. (Extendibility of rays is inherited by subsets with full ideal boundary.) If $\tau \subset \partial_{Tits}X$ is a panel then by 3.8 the action of $Hol(\tau)$ on C_τ by isometries satisfies 4.1 and therefore 4.2 applies. In the case that $\Sigma_\tau\partial_{Tits}X \cong \partial_\infty C_\tau$ is homeomorphic to a sphere, it can be identified with the boundary of a rank-one symmetric space canonically up to conformal diffeomorphism, and $\Sigma_{\phi\tau}X'$ as well. In this situation the "differentials" (26) are conformal diffeomorphisms because they are equivariant with respect to (27). In the second case that $\Sigma_\tau\partial_{Tits}X \cong \partial_\infty C_\tau$ is disconnected, C_τ and $C_{\phi\tau}$ are metric trees and (26) is conformal in the sense that it is induced by a homothety (4.20).

The ideal boundary $\partial_\infty X$, equipped with the cone topology and Tits metric, is a compact topological spherical building. The cone topology can be induced by a metric and this allows us to apply the results from [BS87] on automorphism groups of topological spherical buildings. In particular, [BS87, theorem 2.1] implies:

Theorem 5.9 (Burns-Spatzier) *$Aut(\partial_\infty X)$ is locally compact.*

We denote by F the space of chambers in $\partial_{Tits}X$. The cone topology induces a topology on F which makes F a compact space.

Lemma 5.10 *There exist finitely many chambers $\sigma_1, \dots, \sigma_s$ such that the map*

$$Aut(\partial_\infty X) \longrightarrow F^s \setminus Diag; \phi \mapsto (\phi\sigma_1, \dots, \phi\sigma_s) \tag{28}$$

is proper[14].

Proof: Choose $\sigma_{r+1}, \dots, \sigma_s$ as the chambers of an apartment a, and let $\tau_1, \dots, \tau_r$ be the panels of σ_{r+1}. An automorphism ϕ is determined by its effect on a and the spaces $\Sigma_{\tau_i}\partial_{Tits}X$, because $\partial_{Tits}X$ is the convex hull of the apartment a and all chambers adjacent to its chamber σ_{r+1}[15]. Choose for each panel τ_i a chamber $\sigma_i \not\subset a$ with $\sigma_i \cap \sigma_{r+1} = \tau_i$. Clearly (28) is continuous. Let (ϕ_n) be a sequence in $Aut(\partial_\infty X)$ whose image under (28) is bounded, i.e. does not accumulate at $Diag$. We have to show that (ϕ_n) is bounded, respectively it suffices to show that there is a bounded subsequence. After passing to a subsequence, we may assume that $\phi_n\sigma_i \to \bar{\sigma}_i$ with pairwise different limits $\bar{\sigma}_i$. Denote $\bar{\tau}_i := \lim \phi_n\tau_i$. For each $i \leq r$ the sequence of conformal homeomorphisms $\Sigma_{\tau_i}\partial_{Tits}X \to \Sigma_{\phi_n\tau_i}\partial_{Tits}X$ converges on a triple of points (namely on $\Sigma_{\tau_i}(a \cup \sigma_i)$) to an injective limit map and hence subconverges uniformly to a (conformal) homeomorphism $\Sigma_{\tau_i}\partial_{Tits}X \to \Sigma_{\bar{\tau}_i}\partial_{Tits}X$. It follows that (ϕ_n) subconverges uniformly to a building automorphism. $\square$

Consequence 5.11 *The sequence $(\phi_n) \subset Aut(\partial_\infty X)$ is unbounded iff there exist adjacent chambers σ, σ' such that $\phi_n\sigma$ and $\phi_n\sigma'$ converge in F to the same chamber.*

[14] $Diag$ denotes the *generalized diagonal* consisting of tupels with at least two equal entries.

[15] Proof: The convex hull is a subbuilding B' of maximal dimension [KL96, prop. 3.10.3]. Since any panel is projectively equivalent to a panel τ_i, B' is a neighborhood of $int(\tau)$ for any panel $\tau \subset B'$. We can connect an interior point of any chamber to a point in a by a geodesic avoiding simplices of codimension ≥ 2. It follows that all chambers are contained in B' and $B' = \partial_{Tits}X$.

5.3 The case of no branching

A major part of the arguments in this section follows the lines of Gromov's proof of his Rigidity Theorem [BGS85] and the study of topological spherical buildings in [BS87].

Assumption 5.12 *X is a locally compact Hadamard space with extendible rays and $\partial_{Tits}X$ is a thick irreducible spherical building of dimension $r-1 \geq 1$. Moreover we assume in this section that complete geodesics in X do not branch.*

For every point $x \in X$ there is an involution

$$\iota_x : \partial_\infty X \longrightarrow \partial_\infty X$$

which maps $\xi \in \partial_\infty X$ to the other boundary point of the unique geodesic extending the ray $\overline{x\xi}$.

Lemma 5.13 *$\iota_x \in Aut(\partial_\infty X)$.*

Proof: The absence of branching implies that ι_x is continuous. By 5.6, each ray emanating from x is contained in a maximal flat F. ι_x restricts on the unit sphere $\partial_{Tits}F$ to the antipodal involution. Hence ι_x maps every chamber isometrically to a chamber and is therefore 1-Lipschitz continuous with respect to the Tits distance. The claim follows because $\iota_x^{-1} = \iota_x$. $\square$

5.13 shows that the group $Aut(\partial_\infty X)$ is large. Our aim is to unmask it as the isometry group of a symmetric space. Denote by Inv the subgroup consisting of all products of an even number of involutions ι_{x_i}.

Lemma 5.14 *Inv is path connected and it is contained in every open subgroup of $Aut(\partial_\infty X)$.*

Proof: The map $X \times X \to Aut(\partial_\infty X); (x_1, x_2) \mapsto \iota_{x_1}\iota_{x_2}$ is continuous and hence Inv is path connected. The second assertion follows in view of $(\iota_{x_1}\iota_{x_2})(\iota_{x_2}\iota_{x_2'}) = \iota_{x_1}\iota_{x_2'}$. $\square$

Lemma 5.15 *For any two chambers σ_1, σ_2 in a thick spherical building B there is a common antipodal chamber. Refinement: For any two simplices of the same type*[16] *there is a common antipodal simplex.*

Proof: Let $\hat{\sigma}$ be a chamber antipodal to σ_1 and $\gamma : [0, \pi] \to B$ a unit speed geodesic avoiding codimension-2 faces with $\gamma(0) \in int(\sigma_2)$ and which intersects $int(\hat{\sigma})$. If $\gamma(\pi) \in \hat{\sigma}$ then we are done. Otherwise let $\tau \subset \partial\hat{\sigma}$ be the panel where γ exits $\hat{\sigma}$. Since B is thick, there exists a chamber $\hat{\sigma}'$ opposite to σ_1 so that $\hat{\sigma}' \cap \hat{\sigma} = \tau$. Let $\gamma' : [0, \pi] \to B$ be a unit speed geodesic with $\gamma'(0) = \gamma(0)$, $\dot{\gamma}'(0) = \dot{\gamma}(0)$, which agrees with γ up to $\hat{\sigma}$ and then turns through τ into the interior of $\hat{\sigma}'$. We repeat this procedure until it terminates after finitely steps and yields a chamber opposite to σ_1 and σ_2. The refinement follows directly. $\square$

[16] The *type* of a simplex is its image under the canonical (accordeon) projection to the model Weyl chamber Δ_{model}.

Consequence 5.16 *For any simplex τ, Inv acts transitively on the compact space F_τ of simplices of same type as τ. In particular, Inv acts transitively on the compact space F of Weyl chambers*[17].

Now we investigate the dynamics on $\partial_\infty X$ of elements which correspond to translations (transvections) along geodesics in symmetric spaces.

Lemma 5.17 *Suppose that $\rho : [0,\infty) \to X$ is a ray asymptotic to ξ and that $U \subset \partial_{Tits}X$ is a compact set of ξ-antipodes. Then $\iota_{\rho(t)}U \to \{\xi\}$ as $t \to \infty$.*

Proof: In every $\Sigma_\eta \partial_{Tits}X$, $\eta \in U$, we choose an apartment α_η so that the apartments $\partial_\infty persp_{\eta\xi}\alpha_\eta \subseteq \Sigma_\xi \partial_{Tits}X$ coincide. Consider sequences $t_n \to \infty$ and $(\eta_n) \subset U$. We have to show that $\iota_{\rho(t_n)}\eta_n \to \xi$. Let F_n be a maximal flat containing the ray $\overline{\rho(t_n)\eta_n}$ and satisfying $\Sigma_{\eta_n}\partial_\infty F_n = \alpha_{\eta_n}$.

Sublemma 5.18 *The family of flats F_n is bounded.*

Proof: Assume the contrary and, after passing to a subsequence, that $\eta_n \to \eta \in U$. Denote by a the unique apartment in $\partial_{Tits}X$ containing ξ, η and so that $\Sigma_\eta a = \alpha_\eta$. Let $R > 0$ be large. F_n depends continuously on t_n (by "no branching"), and by decreasing the t_n we can acheive that $d(F_n, \rho(0)) = R$ for almost all n. Still $t_n \to \infty$ if R is chosen sufficiently large; namely $d(\overline{\rho(t)\eta}, \rho(0))$ is bounded because there exists a geodesic asymptotic to ξ and η. The F_n subconverge to a maximal flat F with $d(F, \rho(0)) = R$ and $\partial_\infty F = a$. This can't be possible for arbitrarily large R because the family of flats with ideal boundary a is compact, a contradiction. $\Box$

All flats arising as limits of (F_n) are asymptotic to ξ, η and the antipodes $\iota_{\rho(t_n)}\eta_n$ of η_n in $\partial_\infty F_n$ converge to an antipode of η, i.e. they converge to ξ. $\Box$

Consequence 5.19 *Let $c : \mathbb{R} \to X$ be a geodesic, $\xi_\pm := c(\pm\infty)$ and $a_t := \iota_{c(t)}\iota_{c(-t)}$. Then $\lim_{t\to\infty} a_t\eta = \xi_+$ iff $\angle_{Tits}(\eta, \xi_-) = \pi$. The convergence is uniform on compact sets of ξ_--antipodes.*

Proof: By 5.17, $\iota_{c(-t)}\eta \to \xi_-$ uniformly. Then for large t, $\iota_{c(-t)}\eta$ and ξ_+ are antipodes. Applying 5.17 again yields the claim. $\Box$

Denote by $B(\xi_+, \xi_-) \subset \partial_{Tits}X$ the subbuilding defined as the union of all minimizing geodesics with endpoints $\xi_\pm$, or equivalently, the union of all apartments containing $\xi_\pm$. There is a folding map (building morphism, see [KL96, sec. 3.10]) $fold : \partial_{Tits}X \to B(\xi_+, \xi_-)$ which is uniquely determined by the property that

$$\angle_{Tits}(fold\eta, \xi_-) = \angle_{Tits}(\eta, \xi_-) \qquad \text{and} \qquad \overrightarrow{\xi_-(fold\eta)} = \overrightarrow{\xi_-\eta}$$

for all $\eta \in \partial_{Tits}X$ with $\angle_{Tits}(\eta, \xi_-) < \pi$ and $fold\eta = \xi_+$ if $\angle_{Tits}(\eta, \xi_-) = \pi$.

Refinement 5.20 $\lim_{t\to\infty} a_t = fold$.

[17] F is the analog of Fürstenberg boundary in the symmetric space case.

Proof: By 5.19 and because all a_t fix the Tits neighborhood $B(\xi_+, \xi_-)$ of ξ_- pointwise. $\square$

Proposition 5.21 *$Aut(\partial_\infty X)$ is a semisimple Lie group whose identity component has trivial center.*

Proof: 1. $Aut(\partial_\infty X)$ is a Lie group: Let $G' \subseteq Aut(\partial_\infty X)$ be an open subgroup, c a geodesic, $\xi_\pm = c(\pm\infty)$ and U_+ a neighborhood of ξ_+ which is chosen so small that all points in U_+ with the same Δ_{model}-direction (type) as ξ_+ are ξ_--antipodes (using the lower semicontinuity of Tits distance). Suppose $H \subset G'$ is an invariant subgroup contained in the neighborhood $\{\phi \in G' : \phi\xi_+ \in U_+\}$ of e. Then $H\xi_+$ consists of ξ_--antipodes. Hence $H\xi_+ = a_t H a_t^{-1} \xi_+ = a_t H \xi_+ \to \{\xi_+\}$ as $t \to \infty$, thus $H\xi_+ = \xi_+$. Since $Fix(H)$ is G'-invariant and convex with respect to the Tits metric it follows from 5.16 that $Fix(H) = \partial_\infty X$ and $H = \{e\}$. So there are neighborhoods of the identity in G' which don't contain non-trivial invariant subgroups. 2.20 implies that $Aut(\partial_\infty X)$ is a Lie group.

Sublemma 5.22 *Every non-trivial isometry ϕ of a thick spherical building B different from a sphere carries some point to an antipode.*

Proof: We may assume without loss of generality that B has no spherical join factor. If the assertion were not true then ϕ would be homotopic to the identity and therefore preserve every apartment and hence every simplex, so $\phi = id$. $\square$

2. Semisimplicity: Suppose that A is an invariant abelian subgroup of $Aut_o(\partial_\infty X)$. Let $a \in A$ be a non-trivial element and choose a simplex τ_- such that τ_- and $a\tau_-$ are opposite (using 5.22). τ_- then has involution-invariant[18] type. Let $c : \mathbb{R} \to X$ be a geodesic with $c(-\infty) \in int(\tau_-)$ and τ_+ the simplex containing $c(+\infty)$. Set $a_n := \iota_{c(n)}\iota_{c(-n)} \in Inv$ and $b_n := a_n a a_{-n} \in A$. 5.19 implies $\lim_{n\to\infty} b_n\tau = \tau_+$ for all simplices in the open subset $W = \{\tau \in F_{\tau_-} : \tau \text{ and } \tau_+ \text{ are opposite}\}$ of F_{τ_-}. In view of 5.15, W and the attractor τ_+ are uniquely determined by the dynamics of (b_n) and therefore are preserved by the centralizer of (b_n) in $Aut_o(\partial_\infty X)$. Thus A has fixed points on F_{τ_-}. 5.16 implies that the action of A on F_{τ_-} is trivial. The fixed point set of A on $\partial_{Tits} X$ includes the convex hull of all simplices in F_{τ_-} and this is the whole building $\partial_{Tits} X$ by irreducibility[19]. So $A = \{e\}$. This shows that all abelian invariant subgroups of $Aut_o(\partial_\infty X)$ are trivial, hence also the solvable invariant subgroups. This finishes the proof of 5.21. $\square$

As a consequence of the proposition, there is a symmetric space X_{model} of non-compact type and an isomorphism

$$Aut_o(\partial_\infty X) \xrightarrow{\cong} Isom_o(X_{model}) \tag{29}$$

of Lie groups.

[18] The type of a simplex is *involution-invariant* if its antipodal simplices have the same type, or equivalently, if the type is fixed by the self-isometry of Δ_{model} which is induced by the involution of the spherical Coxeter complex.

[19] The convex hull of simplices of the same involution-invariant type in the spherical Coxeter complex is a subsphere, hence everything by irreducibility.

Lemma 5.23 *The centralizer of every involutive boundary automorphism ι_x is compact.*

Proof: Suppose that (ϕ_n) is an unbounded sequence in the centralizer of ι_x. Then there are adjacent chambers σ, σ' so that $\lim \phi_n \sigma = \lim \phi_n \sigma'$ (by 5.11). The sequence of conformal diffeomorphisms (differentials) $\Sigma_{\sigma\cap\sigma'}\partial_{Tits}X \rightarrow \Sigma_{\phi_n(\sigma\cap\sigma')}\partial_{Tits}X$ is unbounded and converges everywhere except in at most one point to a constant map $\Sigma_{\sigma\cap\sigma'}\partial_{Tits}X \rightarrow \Sigma_{\lim \phi_n(\sigma\cap\sigma')}\partial_{Tits}X$. Denote by $s \subset \partial_{Tits}X$ the wall spanned by the opposite panels $\sigma\cap\sigma'$ and $\iota_x(\sigma\cap\sigma')$. It follows that for all half-apartments $h \subset \partial_{Tits}X$ with $\partial h = s$ with the exception of at most one half-apartment h_0, the limits $\lim \phi_n\big|_h$ exist and have the same half apartment $\bar{h}$ as image. Since $\partial_{Tits}X$ is thick, we find an ι_x-invariant apartment a containing s but not h_0. So $\phi_n\big|_a$ converges to a non-injective map $a \rightarrow \bar{h}$ commuting with ι_x, i.e. sending antipodes to antipodes. Such a map can't exist and we reach a contradiction. $\Box$

Sublemma 5.24 *Let X_0 be an irreducible symmetric space. Every automorphism of $Isom_o(X_0)$ is the conjugation by an isometry, i.e. $Isom(X_0) \cong Aut(Isom_o(X_0))$.*

Proof: $X_0 = G/K$. $\Box$

The involution $\iota_x \in Aut(\partial_\infty X)$ induces by conjugation an involutive automorphism of $Aut_o(\partial_\infty X)$, hence an involutive isomorphism of $Isom_o(X_{model})$ via (29), and as a consequence of 5.23, the corresponding involutive isometry of X_{model} is the reflection at a point $\Phi(x) \in X_{model}$. We obtain a proper continuous map

$$\Phi : X \longrightarrow X_{model}. \tag{30}$$

Another direct consequence is that products $\iota_x\iota_{x'}$ of two involutions correspond to translations (or the identity) in $Isom(X_{model})$. For any flat $F \subset X$ whose ideal boundary $\partial_\infty F$ is a singular sphere we denote by $T_F \subset Aut_o(\partial_\infty X)$ the subset of all $\iota_x\iota_{x'}$ with $x, x' \in F$.

Lemma 5.25 *As a subset of $Isom(X_{model})$, T_F is the group of translations along a flat F^Φ of the same dimension as F. Moreover $rank(X_{model}) = r$.*

Proof: Let $a \subset \partial_{Tits}X$ be an apartment containing $\partial_\infty F$, σ a chamber in a and $\xi_1, \ldots, \xi_r$ the vertices of σ. Moreover denote by τ_i the panel of σ opposite to ξ_i, and by $\hat{\sigma}, \hat{\tau}_i, \xi_i$ the respective antipodal objects in a. An automorphism ϕ of $\partial_{Tits}X$ which fixes a pointwise is determined by its actions on the spaces $\Sigma_{\tau_i}\partial_{Tits}X$. We therefore obtain an embedding

$$Stab_{Aut(\partial_\infty X)}(a) \hookrightarrow \prod_{i=1}^{r} Homeo(\Sigma_{\tau_i}\partial_{Tits}X).$$

An automorphism which fixes the subbuilding $\partial_{Tits}P(\{\xi_i, \hat{\xi}_i\})$ is determined by its action on $\Sigma_{\tau_i}\partial_{Tits}X$ alone and we get an embedding

$$Stab_{Aut(\partial_\infty X)}(\partial_{Tits}P(\{\xi_i, \hat{\xi}_i\})) \hookrightarrow Homeo(\Sigma_{\tau_i}\partial_{Tits}X).$$

Each $\Sigma_{\tau_i}\partial_{Tits}X$ is identified with boundary of a rank-one symmetric space. $\phi \in Stab_{Aut(\partial_\infty X)}(a)$ acts on $\Sigma_{\tau_i}\partial_{Tits}X$ by a conformal diffeomorphism (compare the discussion in section 5.2) which fixes at least the two point set $\Sigma_{\tau_i}a$. This diffeomorphism is hence contained in a subgroup of the conformal group isomorphic to $\mathbb{R} \times cpt$. As a consequence, $Stab_{Aut(\partial_\infty X)}(a)$ topologically embeds into a group $\cong \mathbb{R}^r \times cpt$ and the subgroups $H_i = Stab_{Aut(\partial_\infty X)}(\partial_{Tits}P(\{\xi_i, \hat{\xi}_i\}))$ embed into $\mathbb{R} \times cpt$. Moreover H_i centralises H_j for $i \neq j$. It follows that all translations in $Isom(X_{model})$, which correspond to products $\iota_x\iota_{x'}$ such that x, x' lie on a geodesic asymptotic to ξ_i and $\hat{\xi}_i$, lie in the same 1-parameter subgroup T_i. Moreover the T_i commute with each other. Since $x \mapsto \iota_x$ is proper, the first assertion follows.

If F is a maximal flat with $\partial_\infty F = a$ then the centralizer of T_F is contained in $Stab_{Aut(\partial_\infty X)}(a)$ and thus contains no subgroup $\cong \mathbb{R}^{r+1}$. Hence $rank(X_{model})$ can't be greater than r. $\square$

Consequently, (30) sends maximal flats to maximal flats. Flats whose ideal boundaries are singular spheres arise as intersections of maximal flats and hence go to singular flats. It follows from irreducibility that Φ restricts to a homothety on every flat and clearly the scale factors for restrictions to different flats agree. Since X is geodesically complete by assumption, every pair of points lies in a maximal flat (5.4) and it follows that Φ is a homothety. This concludes the proof of the main result of this section:

Theorem 5.26 *Let X be a locally compact Hadamard space with extendible geodesics and whose Tits boundary is a thick irreducible spherical building of dimension $r-1 \geq 1$. If complete geodesics in X don't branch then X is a Riemannian symmetric space of rank r.*

The argument above also shows that, for an irreducible symmetric space X_0 of rank ≥ 2, the Lie groups $Isom(X_0)$ and $Aut(\partial_\infty X_0)$ have equal dimension and hence the natural embedding $Isom(X_0) \hookrightarrow Aut(\partial_\infty X_0)$ is open and induces an isomorphism of identity components. Of course, more is true:

Theorem 5.27 (Tits) *Let X_0 be an irreducible symmetric space of rank ≥ 2. Then the natural embedding*

$$Isom(X_0) \longrightarrow Aut(\partial_\infty X_0) \tag{31}$$

is an isomorphism.

Proof: Let ψ be an automorphism of $\partial_\infty X_0$. We have to show that ψ is induced by an isometry of X_0. ψ induces an automorphism α of $Aut_o(\partial_\infty X_0) \cong Isom_o(X_0)$ which sends the stabilizer of an apartment a to the stabilizer of ψa, i.e. it sends the group of translations along the flat F_a filling in the apartment a ($\partial_\infty F_a = a$) to the translations along $F_{\psi a}$. The isometry Ψ inducing α (5.24) thus satisfies $\Psi F_a = F_{\psi a}$, i.e. $\partial_\infty \Psi(a) = \psi a$ for all apartments a and it follows $\partial_\infty \Psi = \psi$. $\square$

5.27 implies 1.3 in the smooth case.

5.4 The case of branching

Assumption 5.28 *X is a locally compact Hadamard space with extendible rays and $\partial_{Tits}X$ is a thick irreducible spherical building of dimension $r-1 \geq 1$. Moreover we assume in this section that some complete geodesics branch in X.*

Note that now we can't expect a big group $Aut(\partial_\infty X)$ of boundary automorphisms. There exist completely asymmetric Euclidean buildings of rank 2. Our approach is based on the observation that nevertheless the cross sections of all parallel sets are highly symmetric (3.8).

5.4.1 Disconnectivity of Fürstenberg boundary

The aim of this section is:

Proposition 5.29 *If for some panel σ of $B = \partial_{Tits}X$ the space $\Sigma_\sigma B$ is totally disconnected, then this is true for all panels.*

Proof: We first consider the case when B is one-dimensional. l denotes the length of a Weyl arc and irreducibility implies $\pi/l \geq 3$. The vertices (singular points) of B can be two-coloured, say blue and red, so that adjacent vertices have different colours. The distance of two vertices is an even multiple of l iff they have the same colour.

According to 3.8, the Hadamard spaces C_ξ satisfy 4.1 for all vertices $\xi \in B$. 4.2 tells that $\Sigma_\xi B$ is homeomorphic to a sphere of dimension ≥ 1, a Cantor set or a finite set with at least 3 elements (because B is thick). Vertices $\xi_1, \xi_2 \in B$ of the same colour are projectively equivalent[20] and therefore the spaces of directions $\Sigma_{\xi_i} B$ are homeomorphic. If π/l is odd then any two antipodal vertices have different colours and the $\Sigma_\xi B$ are homeomorphic for all vertices ξ. If π/l is even (and hence ≥ 4 by irreducibility), we have to rule out the possibility that $\Sigma_\xi B$ is disconnected for blue vertices ξ and connected for red vertices. Let us assume that this were the case.

Sublemma 5.30 *If $\Sigma_\xi B$ is a sphere for red vertices ξ then $\Sigma_\eta B$ can't be finite for blue vertices η.*

Proof: Assume that $\Sigma_\xi B$ is a sphere for red vertices ξ and $\Sigma_\eta B$ is finite for blue vertices η.

1. Red vertices ξ, ξ' of distance $4l$ lie in the same path component of the singular set $Sing(B)$: There exists a red vertex η with $d(\xi,\eta) = d(\xi',\eta) = 2l$. ξ, ξ', η lie in an apartment a (because $4l \leq \pi$). Let $\hat\eta$ be the antipode of η in a. Since $\Sigma_\eta B$ is path-connected we can continuously deform the geodesic $\overline{\eta\xi\hat\eta}$ to the geodesic $\overline{\eta\xi'\hat\eta}$, so ξ and ξ' can be connected by a red path.

2. For every red vertex ξ_0 the (red) distance sphere $S_{2l}(\xi_0)$ is path-connected: Let ξ_1, ξ_2 be red vertices with $d(\xi_i, \xi_0) = 2l$. There is a vertex ξ_2' in the same path component of $S_{2l}(\xi_0)$ as ξ_2 such that $d(\xi_1, \xi_2') = 4l$. (Deform as in 1. using an antipode of ξ_0.)

[20] For 1-dimensional spherical buildings *projective equivalence* is the equivalence relation for vertices generated by antipodality.

3. $S_{2l}(\xi_0)$ is a manifold of the same dimension as $\Sigma_{\xi_0}B$: We introduce local coordinates on $S_{2l}(\xi_0)$ near ξ as follows. Let η be the midpoint of $\overline{\xi_0\xi}$, i.e. $d(\xi_0,\eta) = d(\eta,\xi) = l$. Choose antipodes $\hat{\xi}_0$ of ξ_0 and $\hat{\eta}$ of η. For $\xi' \in S_{2l}(\xi_0)$ near ξ the midpoint η' of $\overline{\xi_0\xi'}$ is close to η and $d(\eta',\hat{\eta}) = \pi$, $d(\eta',\hat{\xi}_0) = d(\xi',\hat{\eta}) = \pi - l$. $\overrightarrow{\hat{\xi}_0\eta'}$ and $\overrightarrow{\hat{\eta}\xi'}$ are continuous local coordinates for ξ' and it follows that $S_{2l}(\xi_0)$ is a manifold of the same dimension as $\Sigma_{\xi_0}B$.

4. Since $\Sigma_{\xi_0}B$ embeds into $S_{2l}(\xi_0)$ it follows that $S_{2l}(\xi_0) \cong \Sigma_{\xi_0}B$ via the map $\xi \mapsto \overrightarrow{\xi_0\xi}$, and $S_{2l}(\xi_0)$ is contained in the suspension $B(\xi_0,\hat{\xi}_0)$. This implies that the cardinality of $\Sigma_\eta B$ is 2 and contradicts thickness. $\square$

For the rest of the proof of 5.29 we assume that $\Sigma_\eta B$ is a Cantor set for blue vertices η and $\Sigma_\xi B$ is a sphere for red ξ.

Sublemma 5.31 *Let $\xi,\eta,\eta' \in B$ be distinct vertices (of the same color) with $d(\xi,\eta) = d(\xi,\eta') = \pi - 2l$ and let U be a neighborhood of ξ. Then there exists a vertex $\xi' \in U$ satisfying*

$$d(\xi',\eta) = \pi - 2l \qquad \textit{and} \qquad d(\xi',\eta') = \pi. \tag{32}$$

Proof: Let ζ be the vertex with $\overline{\xi\eta} \cap \overline{\xi\eta'} = \overline{\xi\zeta}$ and ω the vertex on $\overline{\zeta\eta}$ adjacent to ζ. Extend $\overline{\omega\xi}$ beyond ξ to a geodesic $\overline{\omega\hat{\omega}}$ of length π. $\Sigma_\omega B$ has no isolated points and we can pick a geodesic γ connecting ω and $\hat{\omega}$ so that the initial vector $\Sigma_\omega\gamma$ is close to $\overrightarrow{\omega\zeta}$ and the vertex $\xi' \in \gamma$ with $d(\xi',\omega) = d(\xi,\omega)$ lies in U. By construction, (32) holds. $\square$

Sublemma 5.32 *Let $\gamma : (-\epsilon,\epsilon) \to Sing(B)$ be a continuous path in the red singular set and ξ be a red vertex so that $d(\xi,\gamma(0)) = \pi - 2l$. Then $d(\xi,\gamma(t)) = \pi - 2l$ for t close to 0.*

Proof: Let η be a blue vertex adjacent to ξ so that $d(\eta,\gamma(0)) = \pi - l$. The set of vertices at distance $\pi - l$ from η is open in the singular set and so $d(\eta,\gamma(t)) = \pi - l$ for t close to 0. Since $\Sigma_\eta B$ is totally disconnected we have $\overrightarrow{\eta\gamma(t)}=\overrightarrow{\eta\xi}$ for small t, hence the claim holds. $\square$

Sublemma 5.33 *The path γ is constant.*

Proof: It suffices to show that γ is locally constant. There are neighborhoods U of ξ and V of $\eta := \gamma(0)$ so that $d(\xi',\eta') \geq \pi - 2l$ for all vertices $\xi' \in U$ and $\eta' \in V$ (because the Tits distance is upper semicontinuous). We assume without loss of generality that γ does not leave V. Then 5.32 implies that $d(\xi,\gamma(\cdot)) \equiv \pi - 2l$. If γ were not locally constant we could choose t so that $\eta' := \gamma(t) \neq \eta$. Applying 5.31 there exists $\xi' \in U$ so that (32) holds. But 5.32 implies also that $d(\xi',\gamma(\cdot)) \equiv \pi - 2l$. Hence $d(\xi',\eta') = \pi - 2l$, contradicting (32). Thus γ is locally constant. $\square$

Hence, the set of red vertices has trivial path components. But since $\pi/l > 2$, the space of directions $\Sigma_\zeta B$ for any vertex ζ continuously embeds into the blue singular set as well as into the red singular set. Therefore $\Sigma_\zeta B$ can't be connected for any

vertex ζ, contradiction. Hence $\Sigma_\zeta B$ must be a Cantor set for all vertices ζ. This concludes the proof of 5.29 in the 1-dimensional case.

Without much transpiration one can deduce the assertion in the general case $dim(B) \geq 1$: Let σ, τ be panels of the same chamber with angle $\angle(\sigma, \tau) < \pi/2$. Then the 1-dimensional topological spherical building $\Sigma_{\sigma\cap\tau} B$ is irreducible and we have canonical homeomorphisms:

$$\Sigma_\sigma B \cong \Sigma_{\Sigma_{\sigma\cap\tau}\sigma} \Sigma_{\sigma\cap\tau} B, \qquad \Sigma_\tau B \cong \Sigma_{\Sigma_{\sigma\cap\tau}\tau} \Sigma_{\sigma\cap\tau} B,$$

Moreover, $\Sigma_{\sigma\cap\tau} B$ is the ideal boundary of a Hadamard space satisfying 5.28 with $r = 2$, namely of the cross section $CS(f)$ for any $(r-2)$-flat f with $\partial_\infty f \supset \sigma \cap \tau$. Therefore we can apply our assertion in the 1-dimensional case and see that $\Sigma_\sigma B$ is a Cantor set if and only if $\Sigma_\tau B$ is.

Since B is irreducible, for any two panels σ, σ' exists a finite sequence of panels $\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_m = \sigma'$ so that any two successive σ_i are adjacent with angle less than $\pi/2$[21]. This finishes the proof of 5.29. □

5.4.2 The structure of parallel sets

Consider a $(r-1)$-flat $w \subset X$ whose boundary at infinity is a wall in the spherical building $\partial_{Tits} X$. For any panel $\tau \subset \partial_\infty w$, C_τ is canonically isometric to the convex core of $CS(w)$. By 4.2 and 4.16, the following three statements are equivalent:

- $CS(w)$ is the product of a metric tree times a compact Hadamard space.
- $\partial_\infty CS(w)$ is homeomorphic to a Cantor set.
- Some geodesics branch in $CS(w)$[22].

By 5.28 and 5.6, there exists a $(r-1)$-flat w so that $CS(w)$ contains branching geodesics. $\partial_\infty w$ is a wall in $\partial_{Tits} X$ and for any panel $\sigma \subset \partial_\infty w$ we have that $\Sigma_\sigma \partial_{Tits} X \cong \partial_\infty CS(w)$ is a Cantor set. 5.29 implies that $\Sigma_\sigma \partial_{Tits} X$ is a Cantor set for all panels σ in $\partial_{Tits} X$ and hence:

Lemma 5.34 *$\partial_\infty CS(w)$ is tree $\times$ compact for all $(r-1)$-flats w.*

5.4.3 Proof of the main result 1.2

Theorem 5.35 *Let X be a locally compact Hadamard space with extendible rays and whose Tits boundary is a thick irreducible spherical building of dimension $r - 1 \geq 1$. If there are branching complete geodesics in X then X splits as the product of a Euclidean building of rank r times a compact Hadamard space.*

Proof: We first investigate the local structure of X. For every point $x \in X$ we have the canonical 1-Lipschitz continuous projection

$$\theta_x : \partial_{Tits} X \to \Sigma_x X$$

[21] Otherwise we could subdivide the panels of the model Weyl chamber into two families so that panels in different families are orthogonal; this would imply reducibility.

[22] This equivalent to branching of geodesics in C_τ.

which assigns to $\xi \in \partial_{Tits}X$ the direction $\overrightarrow{x\xi} \in \Sigma_x X$. Therefore, if $\xi, \hat{\xi} \in \partial_{Tits}X$ are x-antipodes, i.e. $\angle_x(\xi, \hat{\xi}) = \pi$, then θ_x restricts to an isometry on every geodesic in $\partial_{Tits}X$ of length π connecting ξ and $\hat{\xi}$. By our assumption of extendible rays, every ξ has x-antipodes, and it follows that θ_x restricts to an isometry on every simplex. If $\sigma, \hat{\sigma}$ are open chambers in $\partial_{Tits}X$ which are x-opposite in the sense that there exist x-antipodes $\xi \in \sigma$ and $\hat{\xi} \in \hat{\sigma}$, then θ_x restricts to an isometry on the unique apartment in $\partial_{Tits}X$ containing $\sigma, \hat{\sigma}$ and we call its image an *apartment* in $\Sigma_x X$. If σ_1, σ_2 are open simplices whose θ_x-images intersect then there exists a simplex $\hat{\sigma}$ which is x-opposite to both σ_i. It follows that the θ_x-images of the spheres $span(\sigma_i, \hat{\sigma})$ and therefore the $\theta_x \sigma_i$ coincide. Hence the θ_x-images of open simplices in $\partial_{Tits}X$ are disjoint or they coincide and we call them *simplices* or *faces* in $\Sigma_x X$.

Sublemma 5.36 *The θ_x-images of adjacent chambers $\sigma_1, \sigma_2 \subset \partial_{Tits}X$ are contained in an apartment. (They may coincide.)*

Proof: Let ξ be a point in the open panel $\sigma_1 \cap \sigma_2$ and $\hat{\xi}$ an x-antipode. θ_x is isometric on the half-apartments $h_i = span(\hat{\xi}, \sigma_i)$ because it is isometric on σ_i and $\angle_x(\xi, \hat{\xi}) = \pi$. The union H_i of the rays with initial point x and ideal endpoint $\in h_i$ is a half-r-flat in X. The $(r-1)$-flats ∂H_i coincide and our assumption on cross sections of parallel sets allows two possibilities: Either $H_1 \cup H_2$ is a r-flat and the $\theta_x \sigma_i$ are adjacent chambers in an apartment. Or the $H_1 \cap H_2$ is a non-degenerate flat strip and the $\theta_x \sigma_i$ coincide. $\square$

As a consequence, the centers of adjacent chambers in $\Sigma_x X$ are uniformly separated and the compactness of $\Sigma_x X$ implies that the number of simplices in $\Sigma_x X$ is finite.

Sublemma 5.37 *Any two simplices in $\Sigma_x X$ are contained in an apartment.*

Proof: Since in $\partial_{Tits}X$ any two simplices are contained in an apartment it suffices to prove the following statement: $(*)$ *If c_1 is a chamber contained in an apartment a and c_2 is a chamber so that $c_2 \cap a$ is a panel, then there exists an apartment a' containing the chambers c_1 and c_2.* The rest then follows by induction. To prove $(*)$ we consider the hemisphere $h \subset a$ with $c_1 \subset h$ and $c_2 \cap a \subset \partial h$. Applying 5.36 to the chamber c_2 and the adjacent chamber in h we see that there is a geodesic $\gamma : [0, \pi] \to \Sigma_x X$ contained in $im(\theta_x)$ which starts in $int(c_2)$, passes through $c_2 \cap a$ into h and stays in h for the rest of the time and intersects $int(c_1)$ on its way. The regular endpoints of γ span a unique apartment a' in $\Sigma_x X$, and $a' \supset c_1 \cup c_2$. $\square$

As a consequence, $im(\theta_x)$ is a convex, compact subset of $\Sigma_x X$ and hence itself a CAT(1)-space. The spherical building structure on $\partial_{Tits}X$ induces a spherical building structure on $im(\theta_x)$.

Consequence 5.38 *For any two points $\xi_1, \xi_2 \in \partial_\infty X$ the angle $\angle_x(\xi_1, \xi_2)$ can take only finitely many values which depend on the types $\theta_{\partial_{Tits}X} \xi_i \in \Delta_{model}$.*

Let us denote by Sun_x the union of all rays emanating from x.

Lemma 5.39 *Any two sets Sun_x and Sun_y are disjoint or coincide.*

Proof: Assume that $x \neq y$ and $y \in Sun_x$. We pick an ideal point $\xi \in \partial_\infty X$ and show that the ray $\overline{y\xi}$ is contained in Sun_x: First we extend $\overline{yx}$ to a ray $\overline{yx\eta}$. Then we choose a minimal geodesic connecting $\overrightarrow{yx}$ and $\overrightarrow{y\xi}$ inside $im(\theta_y)$ and extend it beyond $\overrightarrow{y\xi}$ to a geodesic α of length π. Denote the endpoint by u. There is a chamber σ in $\Sigma_y X$ which contains the end of α near u, and we lift σ to a chamber $\tilde{\sigma}$ in $\partial_{Tits} X$. θ_y restricts to an isometry on any apartment $\tilde{a} \subset \partial_{Tits} X$ which contains $\tilde{\sigma}$ and η. Therefore $\tilde{a}$ bounds a flat F which contains x and a ray $\overline{y\xi'}$ with $\overrightarrow{y\xi'} = \overrightarrow{y\xi}$ in $\Sigma_y X$. The rays $\overline{y\xi'}$ and $\overline{y\xi}$ initially coincide (by 5.38) and therefore $Sun_x \cap \overline{y\xi}$ is half-open in $\overline{y\xi}$ towards ξ. Since it is clearly closed, it follows that Sun_x contains the ray $\overline{y\xi}$ and hence $Sun_y \subseteq Sun_x$. Now the segment $\overline{xy}$ is contained in a geodesic. I.e. $x \in Sun_y$ and analoguously $Sun_x \subseteq Sun_y$. This shows that $y \in Sun_x$ iff $Sun_x = Sun_y$. It follows that if $z \in Sun_x \cap Sun_y$ then $Sun_x = Sun_z = Sun_y$, hence the claim. $\square$

It follows that the subsets Sun_x are minimal closed convex with full ideal boundary $\partial_\infty Sun_x = \partial_\infty X$. Consequently they are parallel and, by the second part of 2.2, X decomposes as a product of $Z \times$ *compact*. Z is a geodesically complete Hadamard space and it remains to verify that it carries a Euclidean building structure. Its Tits boundary $\partial_{Tits} Z = \partial_{Tits} X$ and the spaces of directions $\Sigma_z Z$ carry spherical building structures modelled on the same Coxeter complex (S, W) so that the maps $\theta_z : \partial_{Tits} Z \to \Sigma_z Z$ are building morphisms, i.e. they are compatible with the direction maps to the model Weyl chamber Δ_{model}.

$$\theta_{\partial_{Tits} Z} = \theta_{\Sigma_z Z} \circ \theta_z. \tag{33}$$

(The buildings $\Sigma_z Z$ are in general not thick.) Choose a Euclidean r-space E, identify $\partial_{Tits} E \cong S$ and let $W_{aff} \subset Isom(E)$ be the full inverse image of W under the canonical surjection $rot : Isom(E) \to Isom(S)$. Up to isometries in W_{aff} we can pick a canonical chart $E \to F$ for every maximal flat $F \subset Z$. The coordinate changes will be induced by W_{aff}. Since geodesic segments are extendible they are contained in maximal flats and in view of (33) we can assign to them well-defined Δ_{model}-directions. The directions clearly satisfy the angle rigidity property (cf. section 2.3.2) and we hence have a Euclidean building structure on Z modelled on the Euclidean Coxeter complex (E, W_{aff}). (If one wishes, one can reduce the affine Weyl group and obtain a canonical thick Euclidean building structure.) This concludes the proof of 5.35. $\square$

Proof of 1.2: Put in 5.26 and 5.35. Stir gently. $\square$

5.5 Inducing boundary isomorphisms by homotheties: Proof of 1.3

Proof of 1.3: By 1.2, X and X' are symmetric spaces or Euclidean buildings. The Fürstenberg boundary of X is a Cantor set iff X is a Euclidean building. Hence X, X' are either both symmetric or both buildings. The assertion in the symmetric case is the content of 5.27.

We may therefore assume that X and X' are thick irreducible Euclidean buildings of rank $r \geq 2$. Then for any flat f the cross section $CS(f)$ is a Euclidean building

of rank $r - \dim(f)$, and has no Euclidean factor if f is singular. For all geodesics l the canonical embeddings $CS(l) \hookrightarrow X_{l(\pm\infty)}$ of cross sections into spaces of strong asymptote classes are now surjective isometries, and for every $\xi \in \partial_\infty X$, $C_\xi \cong X_\xi$ is a Euclidean building of rank $r-1$ which splitts off a Euclidean de Rham factor of dimension $\dim(\tau_\xi)$ where τ_ξ denotes the simplex containing ξ as interior point. In particular, for all panels $\tau \subset \partial_{Tits} X$, C_τ is a metric tree. As explained in section 5.2, the differentials (26) of (1) are boundary maps of homotheties

$$C_\tau \longrightarrow C_{\phi\tau} \tag{34}$$

and these commute with the system of natural perspectivity identifications (16). The assertion of 1.3 follows if we can pin down every vertex of X by data at infinity. This is acheived by the following **bowtie construction** suggested by Bruce Kleiner: A *bowtie* $\bowtie$ consists of a pair of opposite chambers $\sigma_\bowtie$ and $\hat\sigma_\bowtie$, of vertices $y_i \in C_{\tau_i}$ for each panel $\tau_i \subset \sigma_\bowtie$ and vertices $\hat y_i \in C_{\hat\tau_i}$ for the opposite panels $\hat\tau_i$ so that $persp_{\tau_i\hat\tau_i} y_i = \hat y_i$ holds. $\bowtie$ determines a vertex in X as follows: $\sigma_\bowtie$ and $\hat\sigma_\bowtie$ are contained in the ideal boundary of a unique maximal flat $F_\bowtie \subset X$ and every pair $y_i, \hat y_i$ determines a wall $w_i \subset F_\bowtie$. The r walls w_i intersect in a unique vertex $x_\bowtie$. We say loosely that $\bowtie$ is *contained* in the flat $F_\bowtie$. We call two bowties $\bowtie$ and $\bowtie'$ pre-adjacent if $\sigma_\bowtie \cap \sigma'_\bowtie = \tau_r$, $\hat\sigma_\bowtie = \hat\sigma'_\bowtie$ and $\hat y_i = \hat y'_i$ for all i. (Then also $y_r = y'_r$ holds.) There is an obvious involution on the space of bowties and an equally obvious action of the permutation group S_r and we call two bowties *adjacent* if they are pre-adjacent modulo these operations. Adjacent bowties determine the same vertex. Adjacency spans an equivalence relation on the set of bowties which we denote by "$\sim$".

Lemma 5.40 *$\bowtie \sim \bowtie'$ iff $x_\bowtie = x_{\bowtie'}$.*

Proof: Clearly $\bowtie \sim \bowtie'$ implies $x_\bowtie = x_{\bowtie'}$. To prove the converse, let us assume that $x_\bowtie = x_{\bowtie'}$. We start with a special case:

Sublemma 5.41 *If $\bowtie$ and $\bowtie'$ lie in the same apartment then $\bowtie \sim \bowtie'$.*

Proof: It is enough to check the case when $\sigma_\bowtie$ and $\sigma_{\bowtie'}$ share a panel, i.e. without loss of generality $\tau_1 = \tau'_1$, $\hat\tau_1 = \hat\tau'_1$, $y_1 = y'_1$ and $\hat y_1 = \hat y'_1$. Since $y_1, \hat y_1$ are vertices there exists a half-r-flat $H \subset X$ so that $H \cap F = \partial H = w_1$. If $\bowtie''$ is adjacent to $\bowtie$ and $\bowtie'''$ is adjacent to $\bowtie'$ so that $\sigma_{\bowtie''} = \sigma_{\bowtie'''} \subset \partial_\infty H$ then $\bowtie''$ and $\bowtie'''$ are adjacent. So $\bowtie \sim \bowtie'$. $\square$

Sublemma 5.42 *Let $\hat\sigma$ be a chamber in $\partial_{Tits} X$. Then there exists a bowtie $\bowtie'' \sim \bowtie$ so that $\hat\sigma_{\bowtie''} = \hat\sigma$.*

Proof: It is enough to treat the case when $\hat\sigma$ is adjacent to $\hat\sigma_\bowtie$ in $\partial_{Tits} X$. After replacing $\bowtie$ by an equivalent bowtie (e.g. contained in the same maximal flat) we may assume that $\theta_{x_\bowtie}\hat\sigma$ and $\theta_{x_\bowtie}\sigma_\bowtie$ are opposite chambers in $\Sigma_{x_\bowtie} X$. Then we can choose $\bowtie''$ adjacent to $\bowtie$. $\square$

We refine the previous sublemma:

Sublemma 5.43 *Let F be a maximal flat and $\bowtie$ a bowtie with $\sigma_{\bowtie} \subset \partial_\infty F$ and $x_{\bowtie} \in F$. Let $\hat\sigma$ be any chamber in $\partial_{Tits}X$. Then there exists another bowtie $\bowtie'' \sim \bowtie$ so that $\hat\sigma_{\bowtie''} = \hat\sigma$ and $\sigma_{\bowtie''} \subset \partial_\infty F$.*

Proof: Again, we may assume without loss of generality that the chamber $\hat\sigma$ is adjacent to $\hat\sigma_{\bowtie}$. If $\theta_{x_{\bowtie}}\hat\sigma$ is opposite to the chamber $\theta_{x_{\bowtie}}\sigma_{\bowtie}$ in $\Sigma_{x_{\bowtie}}X$ then we can choose $\bowtie''$ adjacent to $\bowtie$. Otherwise let $\sigma \subset \partial_\infty F$ be the chamber adjacent to $\sigma_{\bowtie}$ so that $\theta_{x_{\bowtie}}\sigma$ is opposite to $\theta_{x_{\bowtie}}\hat\sigma$ and denote by $\bowtie''$ the bowtie with $\sigma_{\bowtie''} = \sigma$, $\hat\sigma_{\bowtie''} = \hat\sigma$ and $x_{\bowtie''} = x_{\bowtie}$. Then $\bowtie''$ is equivalent to a bowtie contained in $F_{\bowtie}$ and hence to $\bowtie$. $\square$

To finish the proof of 5.40 we can first replace $\bowtie'$ by an equivalent bowtie so that $\sigma_{\bowtie} = \sigma_{\bowtie'}$ (5.42) and then replace it in a second step so that $\bowtie'$ and $\bowtie$ lie in the same apartment (5.43). Hence $\bowtie'$ and $\bowtie$ are equivalent (5.41). $\square$

It follows that equivalence classes of bowties in X correspond to vertices. Since (1) induces a map between the spaces of bowties in X and X' which preserves the equivalence relation "$\sim$", it thereby induces a map $\Phi : Vert(X) \to Vert(X')$ on vertices. Φ maps all vertices in a singular flat $f \subset X$ to the vertices of a singular flat $f^\Phi \subset X'$ so that $\phi(\partial_\infty f) = \partial_\infty f^\Phi$. Since X and X' are irreducible buildings, Φ extends to a homothety $\Phi : X \to X'$ and $\partial_\infty\Phi = \phi$. This concludes the proof of 1.3. $\square$

5.6 Extension of Mostow and Prasad Rigidity to singular spaces of nonpositive curvature: Proof of 1.5

Proof of 1.5: We argue as Mostow [Mos73]. A Γ-*periodic* flat is a maximal flat whose stabilizer in Γ acts cocompactly. Due to results of Borel and Ballmann-Brin, Γ-periodic flats lie dense in the space of all flats in X_{model}. By our assumption, there is a Γ-equivariant continuous map

$$\Phi : X_{model} \longrightarrow X.$$

It is a quasi-isometry and carries Γ-periodic flats in X_{model} to Γ-periodic quasi-flats in X with uniform quasi-isometry constants. If a quasi-flat is Hausdorff close to a flat then it lies in a tubular neighborhood of this flat whose radius is uniformly bounded in terms of the quasi-isometry constants ([Mos73, Lemma 13.2] for symmetric spaces and [KL96] for buildings). Density and uniformity imply that Φ maps every flat in X_{model} uniformly close to a flat in X and with this information one can construct a Γ-equivariant boundary isomorphism

$$\Phi_\infty : \partial_\infty X_{model} \longrightarrow \partial_\infty X.$$

By 1.2 X is a symmetric space or Euclidean building, and by 1.3, after suitably rescaling the irreducible factors of X_{model}, Φ_∞ is induced by a Γ-equivariant isometry $X_{model} \to X$. $\square$

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